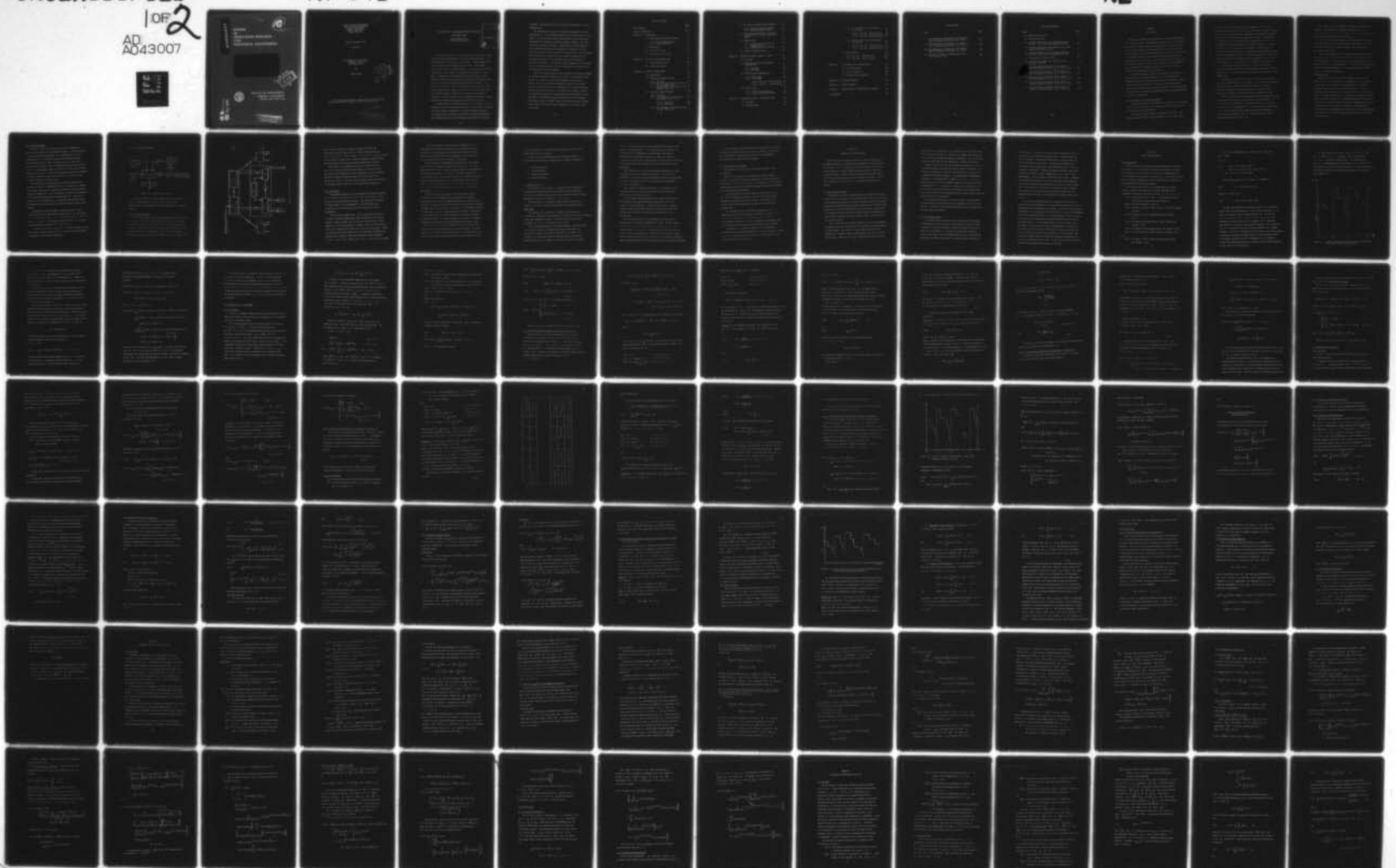
AD-A043 007 CORNELL UNIV ITHACA N Y SCHOOL OF OPERATIONS RESEARC--ETC F/G 15/5
AN ANALYSIS OF A TWO-ECHELON INVENTORY SYSTEM FOR RECOVERABLE I--ETC(U)
JUL 77 K SHANKER N00014-75-C-1172

UNCLASSIFIED TR-341 NL

1 OF 2

AD
A043007



SCHOOL
OF
OPERATIONS RESEARCH
AND
INDUSTRIAL ENGINEERING

COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853

SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK

8

TECHNICAL REPORT NO. 341

July 1977

AN ANALYSIS OF A TWO-ECHELON INVENTORY SYSTEM FOR RECOVERABLE ITEMS

by

Kripa Shanker

This research was partially supported by the Office of Naval Research under Contract N00014-75-C-1172, Task NR 042-335.

AN ANALYSIS OF A TWO-ECHELON INVENTORY SYSTEM FOR RECOVERABLE ITEMS

Kripa Shanker, Ph.D.
Cornell University 1977

In this dissertation, we present an analysis of continuous review models of a two-echelon inventory system for recoverable items. The system consists of a depot and a set of bases. Primary demands occur at the bases for one or several units at a time. It is assumed that demands arrive in a Poisson manner. Upon arrival of a demand for certain units, a like number of failed units are turned in at the base. An inspection of the failed units is carried out to decide whether the units will be repaired at the base or at the depot or will be removed from the system in case repair is not economical. The bases use an $(s-1, s)$ policy for procurement of serviceable units from the depot, and the depot uses an (s,S) policy to procure from the external supplier. Demands in an out-of-stock situation are backlogged. It is assumed that all the locations have infinite repair capacities and repair and procurement lead times are constant.

A common problem in inventory management is to specify the policy parameters that will minimize expected cost per unit time for operating the system subject to constraints of certain performance measures. To formulate such a problem we must find the stationary distributions for inventory position, on-hand inventory, backorders and in-repair

inventory. Our main objective is to find exact expressions for these distributions.

The investigation begins with an extensive analysis of a single location system. The procurement policy is a continuous review (s,S) policy. The inter-arrival times between successive requisitions are independent and identically distributed random variables. The system experiences two types of demands - recoverable and non-recoverable. The two demand processes may be independent or dependent. For the inspection of failed units, two models - batch and unit - are considered. In the batch model, the entire batch is either recoverable or non-recoverable, whereas, in the unit model each unit in a batch is inspected independently. The special cases of compound Poisson demands, (s, nQ) procurement policy, complete recoverability and cmmplete non-recoverability are also considered.

For the two-echelon system we first consider the case where demands at the bases occur for a single unit at a time. The approach is then applied to a general situation where demands at the bases are random. Both the batch and unit inspection models are considered. For the case when there are no condemnations of the item, results are compared with the METRIC model. The METRIC model provides a simple but approximate expression for the probability distribution of system backorders. The comparison indicates that there is a considerable discrepancy between the METRIC results and our results when the depot spare stock is low or when a major proportion of the repair is done at the depot.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

CHAPTER I

INTRODUCTION

This study is devoted to the analysis of recoverable (repairable) item inventory systems. Upon failure, a recoverable item is returned to the source of supply (inventory point) where a decision is made either to remove (condemn) the item from the inventory system or to perform repair on it in order to restore it to a serviceable condition. The decision to repair or condemn the failed item is based on the degree and the nature of failure, the repair facilities available, and the economics involved. Once an item is designated as recoverable, it is presumably more economical to repair the item than it is to dispose of it and replace it with a new item.

Most inventory systems consist of consumable (non-recoverable) items that are predominantly low in cost. In many large-scale industrial activities, military organizations, for example, a large proportion of the inventory investment is in recoverable items, although precentage-wise most items are consumable. Hence management of recoverable item inventory systems from both the design and control viewpoints is important.

1.1 Recoverable Item Inventory Systems

A typical recoverable item inventory system consists of customers, inspection and repair facilities, supply (inventory) points, and an external supplier (manufacturer).

Customers generate the primary demands on the system. While placing requisitions for the replacement of one or several units,

customers turn in a like number of failed units. In view of the description presented earlier, the system experiences two types of demands: recoverable and non-recoverable. Depending on the processes generating the failures, the two demand processes can be either (1) independent or (2) dependent. Demand processes are independent when there are two independent processes generating the two types of failures. Although dependence of the demand processes may arise in different ways, we shall limit the consideration to the case where a single failure process results in both types of failures. Thus for independent demand processes, there are two types of customers from two independent sources whereas for dependent demand processes customers arrive from a single source.

We assume that customers arrive from an infinite population. Also, the inter-arrival times of the customers and the number of units demanded upon an arrival are assumed to be random variables with finite means (known).

Upon arrival, a batch of failed units is inspected to classify the units as repairable or non-repairable. It is assumed that inspection takes a negligible amount of time and the probability of failed units being repairable is known and is the same for all arrivals. We will not consider the decision rules for classifying the items.

After inspection, the repairable units are sent to repair facilities where repairs are performed on a first-come, first-served basis. We shall consider only continuous repair process; that is, no batching is allowed at the repair facilities. It is assumed that repaired units behave exactly like new ones in their performance characteristics. Upon completion of repairs, the unit immediately joins the stock of serviceable units at the supply point.

Supply points stock the ready-for-issue units to resupply the customers. They receive their supplies from two sources: the repair facilities and the external supplier. It is assumed that supply points can stock an unlimited number of units.

An external supplier can supply an unlimited number of units to the system within a known duration of time. This duration, known as procurement lead time, is assumed to be the same for all orders independent of the size.

Thus the main functions of a recoverable item inventory system, in general, are to fill the customer demands, to diagnose (inspect) the failed units, to repair the failed but recoverable units, and to procure units from an external supplier. We have already described the inspection and repair functions. We shall consider the following policies for supply and procurement.

The Supply Policy:

Demands are satisfied from the ready-for-issue stock (also referred to as on-hand inventory) at a supply point. Upon arrival of a requisition, the customer is immediately shipped the quantity requested if there is sufficient on-hand inventory. If there are not enough units on hand, then all the units in stock are dispatched while the balance is backlogged. In either case, the batch of failed units is sent for inspection. The backlogged demands are satisfied on a first-come, first-served basis when the next supply arrives from an external supplier or from repair facilities. The assumption of backlogging may not be appropriate in some cases, but is representative of situations such as military organizations where a captive market situation exists.

The Procurement Policy:

In order to make up for system losses due to condemnations, new units are procured from the external supplier. We shall consider a continuous review reorder-point (s), reorder-level (S) procurement policy. The policy is based on the inventory position which is defined as the sum of the units on hand, on order and in repair minus backorders. When the inventory position drops below the level s, a procurement order is placed so as to bring the inventory position to the level S. Thus the procurement decisions are cognizant not only of those serviceable units on hand or on order, but also of those in repair or awaiting repair.

The main reason for considering the continuous review (s,S) policy is to deal with situations in which the presence of a computerized control system makes it possible to update the inventory levels after each transaction. Also, this policy is known [7] to be superior to the popular continuous review reorder-point (r), fixed order quantity (Q) procurement policy in terms of total reduced costs of procurement and carrying inventory, especially when the order size is random.

In addition, we shall consider a continuous review (s,nQ) policy because of its mathematical simplicity [19]. In this policy, nQ units are ordered each time the inventory position drops below the level s, where n is the largest integer such that the subsequent inventory position is between s + 1 and s + Q.

To provide a better understanding of two-echelon systems, we shall first study a single location system. An outline of the two systems is presented in the following subsections.

1.1.1 Single Location System

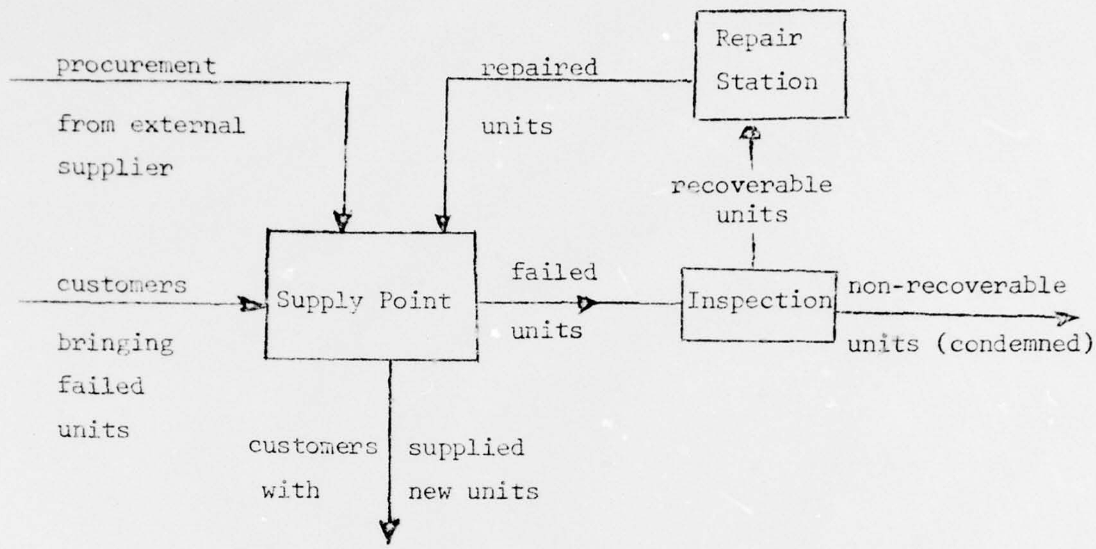


Figure 1.1: Single Location System.

Figure 1.1 shows the schematic diagram of a single location system. System demands are generated at one single point; procurements at the supply point are made directly from the external supplier.

1.1.2 Two-Echelon System.

We shall investigate a two - echelon system as depicted in Figure 1.2. System demands are generated at several locations called bases, which in turn receive their supplies from a central location called a depot. The depot and the bases are also called the upper and lower echelon of the system, respectively. Each location in addition to being an item stocking point, has facilities to perform repairs. The

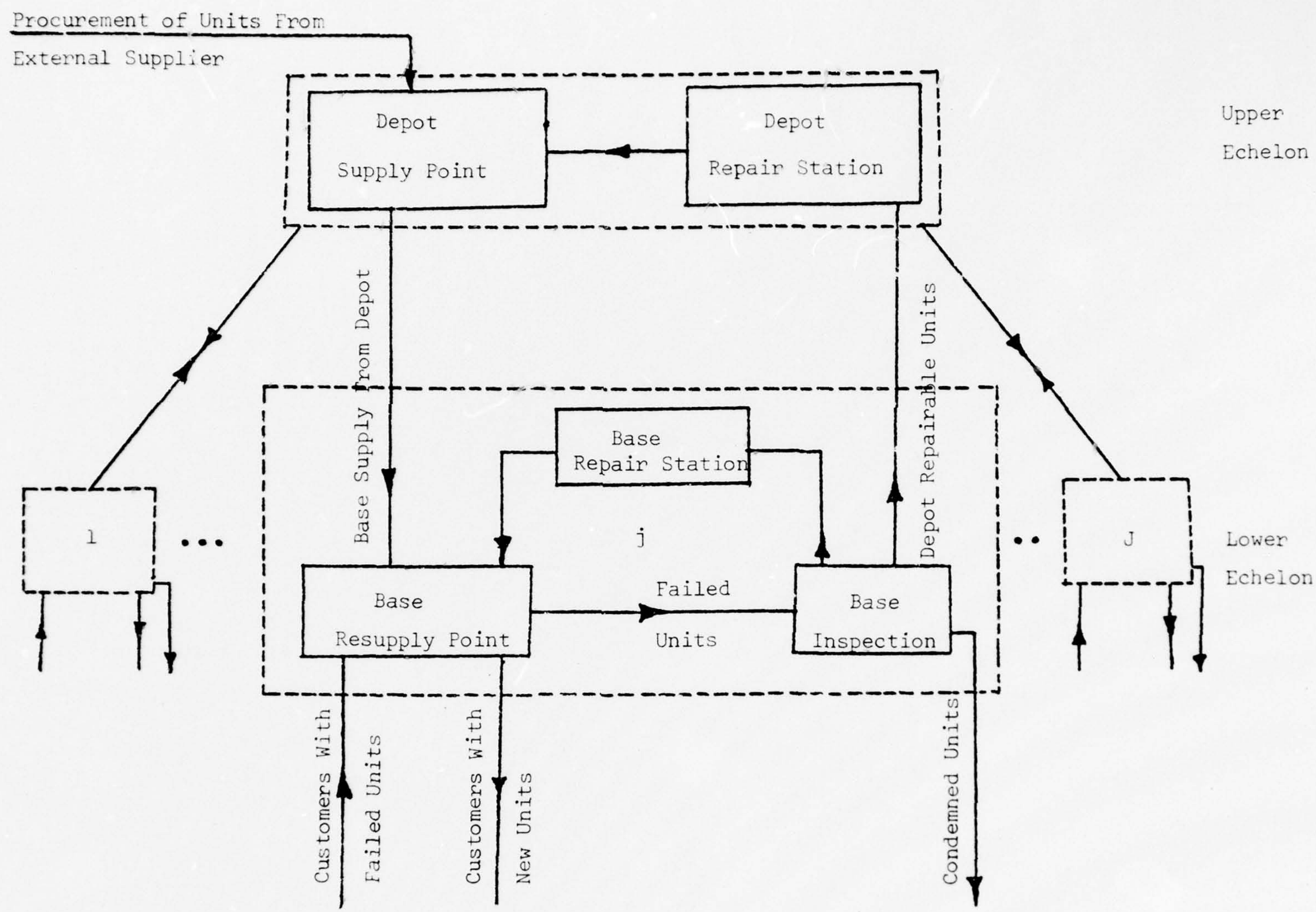


Figure 1.2 Two-Echelon System

depot stock is used only to resupply the bases, and bases are resupplied only by the depot; that is, lateral resupply among the bases is not allowed. Thus procurement of units from the external supplier is done through the depot only. From the viewpoint of network theory, the system looks like a parallel activity arborescence structure.

Upon arrival of requisitions at a base, the failed units are sent to the inspection facility whose function is two fold: first, the units must be classified as either recoverable or non-recoverable, and then if repair is warranted where it will take place at the base or at the depot. The latter decision depends only on the severity of the damage caused to the units and the base repair capability.

1.2. The Problem

The management problem in both the single and two-echelon systems is to establish the operating rules that will minimize expected cost per unit time for operating the system. The solution to the problem is usually sought in the environment of a limited budget and a set of goals to reach certain levels of some measures of system performance.

The major cost components are: cost of acquisition per unit of the item, a fixed procurement setup cost independent of the quantity ordered, a fixed backorder cost each time a stockout occurs, a time-weighted cost for each backorder, a holding cost for the units held in stock, and a charge for the units held at the repair facilities. Given the repair policy and the number of repair facilities, specifying the operating rules includes determining procurement policy parameters s and S for an (s,S) policy, and s and Q for an (s,nQ) policy.

Since all recoverable failures must be repaired and all non-recoverable failures must be replaced at costs independent of the procurement policy parameters, we can ignore both repair costs and acquisition costs (other than fixed procurement or set up costs) as far as determination of these parameters is concerned.

Several measures of effectiveness have been used for inventory systems. Following Hadley and Whitin [7], some of these are: the probability of no stock on hand, the expected number of backorders and expected on-hand inventory at any time. Feeny and Sherbrooke [4] considered fill rate, service rate, ready rate and operational rate as measures of effectiveness for a base stockage system with no condemnations. For a two-echelon system with no condemnation, Sherbrooke [17] found the expected backorders to be the most suitable measure.

The Approach:

In order to solve the problems as described above, several approaches have been suggested [3], viz.expected cost analysis, stationary process analysis, dynamic programming and dynamic process analysis. We shall use an approach based on stationary process analysis. This approach is more appealing to us since it is applicable to general situations and it is also computationally less complex. The approach uses techniques based upon Markov processes and elements of renewal and queueing theory. The principal problem is to find the stationary probability distributions for several stochastic processes. These distributions, if they exist, are functions of the procurement policy used and of the demand distribution, but not of any costs. The cost structure to represent the objective function - expected cost per time unit - can be constructed using these stationary distributions. Also,

the various measures of system effectiveness described earlier can be obtained from these distributions.

In order to obtain expressions for the objective function and system performance criteria, we must find the stationary distributions of:

1. Inventory position
2. On-hand inventory
3. Number of backorders
4. In-repair inventory.

1.3 Scope of the Study

The objective of this study is to obtain an exact expression for the stationary distributions of the stochastic processes mentioned in Section 1.2 for the single location and two-echelon inventory systems described in Sections 1.1.1 and 1.1.2 , respectively.

With reference to the dependent demand processes mentioned in Section 1.1, we shall consider the following two inspection policies.

Batch Model:

Upon arrival, the entire batch is either recoverable or non-recoverable. Inspections are considered to be repeated independent Bernoulli trials with probability r_B(say) of a batch being recoverable and probability $(1 - r_B)$ of it being non-recoverable.

From a practical viewpoint, this model represents situations where the units of a batch fail simultaneously for the same reason and the extent of damage is the same for all units in the batch. For instance, the maintenance system of aircraft engines considered by Muckstadt [11] in his MOD-METRIC model can be described by the batch

model. The engine consists of several modules which contain a large number of recoverable units. In a failed engine, all the units in a module (batch) are considered to have sustained same extent of damage as far as maintenance (repairs) is concerned. Here, the inspection is carried out to decide whether the module will be repaired at the supply point (base) or will be sent to the depot for repair.

Unit Model:

Each failed unit in a batch is inspected independently to determine whether it will be repaired or condemned. Inspections are considered to be repeated independent Bernoulli trials with probability r_U (say) of sending a unit to the repair cycle and probability $(1 - r_U)$ of condemning it.

From a practical viewpoint, this model can be applied to the situations where units failed under different conditions but are submitted in a batch for replacement.

For the single location system, we will consider both the cases of independent and dependent demand processes under (s,S) and (s,nQ) procurement policies. A general structure will be provided to obtain the stationary distributions for the cases of finite and infinite number of repair facilities and general repair time distributions.

The two-echelon sytem will be analyzed for the case where a Poisson process generates the demands at the bases. We will consider a one-for-one $(s-1, s)$ procurement policy at the bases and (s,S) (s,nQ) policies at the depot. The repair times at the depot and the bases as well as procurement lead times between a base and the depot, and the external supplier will be assumed to be deterministic and known. Also, we shall assume that the repair capacity at all locations is infinite.

For both systems we will first obtain the results for the general case where an item will either be repaired or condemned; results will then be derived for the special cases where (i) no condemnations occur and (ii) all failed units are condemned.

1.4 Organization of the Study

We begin with a brief review of the work of previous authors in Chapter II.

Chapter III is devoted to the study of the single location system. We identify the inventory position as a semi-Markov process and obtain its stationary distribution. Following this, the stationary distribution of the stochastic process representing both the backorders and on-hand inventory is obtained.

Both independent and dependent demand processes under (s,S) and (s,nQ) policies are examined. The case of a Poisson process generating the failures is studied in depth. At the end of the chapter, some long-run averages are derived from the stationary distributions.

The two-echelon system is studied in chapters IV and V. In Chapter IV, the case of unit demand at the bases is examined. The case of an arbitrary order size distribution at the bases is considered in Chapter V. In both the chapters, the special cases of no condemnation and complete non-recoverability are considered.

In Chapter VI, we assess the degree to which Sherbrooke's results [17] can serve as approximations to our exact results. Finally, Chapter VII contains concluding comments. In this chapter, we also indicate areas for future research.

CHAPTER II

REVIEW OF SOME PREVIOUS WORK

Most of the vast literature on inventory theory is addressed to consumable items and it is not directly applicable to inventory systems of recoverable items. We shall present a brief review of the work of previous authors for single location and multi-echelon systems for recoverable items. We will set aside the numerous simulation models that have been developed and applied to specific situations. Also, we will concentrate on work dealing with stochastic models for these systems.

2.1 Single Location System

Considerable attention has been devoted to the analysis and design of this problem as a whole and to its subproblems. Feeney and Sherbrooke [5] investigated stochastic recoverable item models that assume compound Poisson demand distributions and complete recoverability of failed items. Schrady [15] examined a deterministic model that permits condemnations. In a subsequent survey report [16], Schrady described approximate solutions to both continuous and periodic review models, and continuous as well as batch repairs.

Allen and D'Esopo [1] allowing condemnations obtained approximate stationary results for expected number of backorders and other measures of effectiveness. They assumed a Poisson demand distribution and deterministic (positive) repair and procurement lead times. In a subsequent paper, Simon and D'Esopo [21] obtained exact results for the

same model with a relaxation of the assumption made in [1] that Poisson process generate the failures. They, however, assumed that recoverable and non-recoverable demand processes are independent. In all the above references repair facility was treated as an infinite server queue.

François Lureau [10] viewed the problem as a queueing process and obtained stationary results for the expected number of backorders. In addition, he obtained the stationary distribution for waiting time of a customer before being resupplied. Assuming complete recoverability throughout, he developed models allowing general repair time distributions and finite as well as infinite number of the repair facilities. Recently, Richards [13] examined the problem with condemnations. His results allow for random lead time and alternate repair disciplines. He, however, maintained the assumption that recoverable and non-recoverable demand processes are independent.

Our analysis is more general in that it permits the recoverable and non-recoverable demand processes to be dependent as well as independent. In addition, for dependent demand processes we consider batch and unit inspection policies.

2.2 Two-Echelon System

A fundamental work on the two-echelon system was the development of METRIC (Multi-Echelon Technique for Recoverable Inventory Control) by Sherbrooke [17] for a completely conservative system that does not allow item condemnation. He considered the problem of allocating several units among a depot and several bases in order to minimize the total expected number of backorders at bases within the limitation of

a given budget. He assumed a compound Poisson demand distribution at each base and a one-for-one procurement policy. Depending upon the nature of failures, repairs could be performed at the base where the demand originated or at the depot. The resulting expressions are approximate, and a special case of our model can be used to check the accuracy of his results. Sherbrooke assumed an arbitrary repair time distribution though his results depend only upon the means of these distributions. We shall assume deterministic repair times. Sherbrooke also presented an approximate method for including item condemnation in his model, assuming that procurements are made on a periodic review basis. For Sherbrooke's model of the conservative system, Muckstadt [12] developed a computationally more efficient approach than the previous work by Sherbrooke, for determining the optimal system stock levels.

A variation of the METRIC model was introduced by Simon [20] to obtain the exact expressions for the stationary distributions of on-hand inventory and of the backorders at the bases. This model is more general than METRIC in that it permits non-recoverability as well as recoverability with positive condemnation rates. It is less general in that all repair and lead times are deterministic and demand distributions at the bases are simple Poisson. In a subsequent comment on Simon's paper, Kruse and Kaplan [9] pointed out that Simon's derivations were valid for the two special cases in which non base-repairable failures are either all depot-repairable or all non-depot repairable. For Simon's model, they suggested a simpler method of deriving the probability expressions for the number of backorders at the bases. We use the same approach for the case of arbitrary demand distributions at the bases.

CHAPTER III

SINGLE LOCATION SYSTEM

3.1 Introduction

In this chapter we study the single location system as described in Section 1.1.1. The analysis, in addition to offering the solution to this system, is also applicable for the upper echelon (depot) of the two-echelon system discussed in Section 1.1.2. The results can also be used to find an approximate solution for an individual location in lower echelons of a multi-echelon system.

The stationary distributions of inventory position, on-hand inventory, number of backorders and in-repair inventory will be obtained for the problem described in Section 1.2. For $t \geq 0$, let

$X(t)$ = the inventory level at time t which consists of the units ready for issue minus any backorders,

$Q(t)$ = in-repair inventory at time t,

$O(t)$ = the number of units on order at time t from the external supplier,

$D_c(t)$ = the number of units condemned during the interval $(0,t]$,

$D_r(t)$ = the number of recoverable units turned in during the interval $(0,t]$,

$C(t)$ = the number of units repaired during the interval $(0,t]$,

$OQ(t)$ = the number of units ordered during the interval $(0,t]$

and

$RQ(t)$ = the number of units received via procurement during the interval $(0,t]$.

Also, for any stochastic process $\{P(t), t \geq 0\}$, $P(t_1, t_2) = P(t_2) - P(t_1^+)$.

Obviously,

$$(3.1) \qquad \begin{aligned} Q(t) &= Q(0) + D_r(t) - C(t), \\ X(t) &= X(0) + C(t) + RQ(t) - D_r(t) - D_c(t), \\ \text{and} \quad O(t) &= O(0) + OQ(t) - RQ(t). \end{aligned}$$

The inventory position $Z(t)$ at time t is defined as

$$(3.2) \qquad Z(t) = X(t) + Q(t) + O(t).$$

From Eqs. (3.1) and (3.2)

$$(3.3) \qquad Z(t) = Z(0) + OQ(t) - D_c(t).$$

We shall first consider the continuous review (s,S) procurement policy. When the inventory position $Z(t)$ falls to the level s or below, a procurement order is placed to bring $Z(t)$ to the level S $(\geq s)$. Upon completion of a repair, $Q(t)$ decreases and $X(t)$ increases by the same amount; thus no change in $Z(t)$. Similarly, upon an arrival of supply from the external supplier, $O(t)$ decreases and $X(t)$ increases by the same amount and therefore $Z(t)$ remains unchanged. Thus $Z(t)$ may change only at the epochs of customer arrivals. Consequently, procurement orders may only be placed at demand epochs. By convention, whenever an order is triggered, $Z(t)$ is meant to include the demand just arrived plus the order that the demand triggered so that $Z(t) = S$ at such epochs.

The state space for the process $\{Z(t), t \geq 0\}$ is the finite set $E = \{i | s+1 \leq i \leq S, \; i \text{ integers}\}$. $Z(t)$ bounces between $s+1$ and S during each ordering cycle. The ordering epochs $R_1, R_2, \ldots,$ are regeneration points for $\{Z(t), t \geq 0\}$ since at each such epoch the inventory position restarts at S and the continuation of the process thereon is a probabilistic replica of the previous cycle (Figure 3.1).

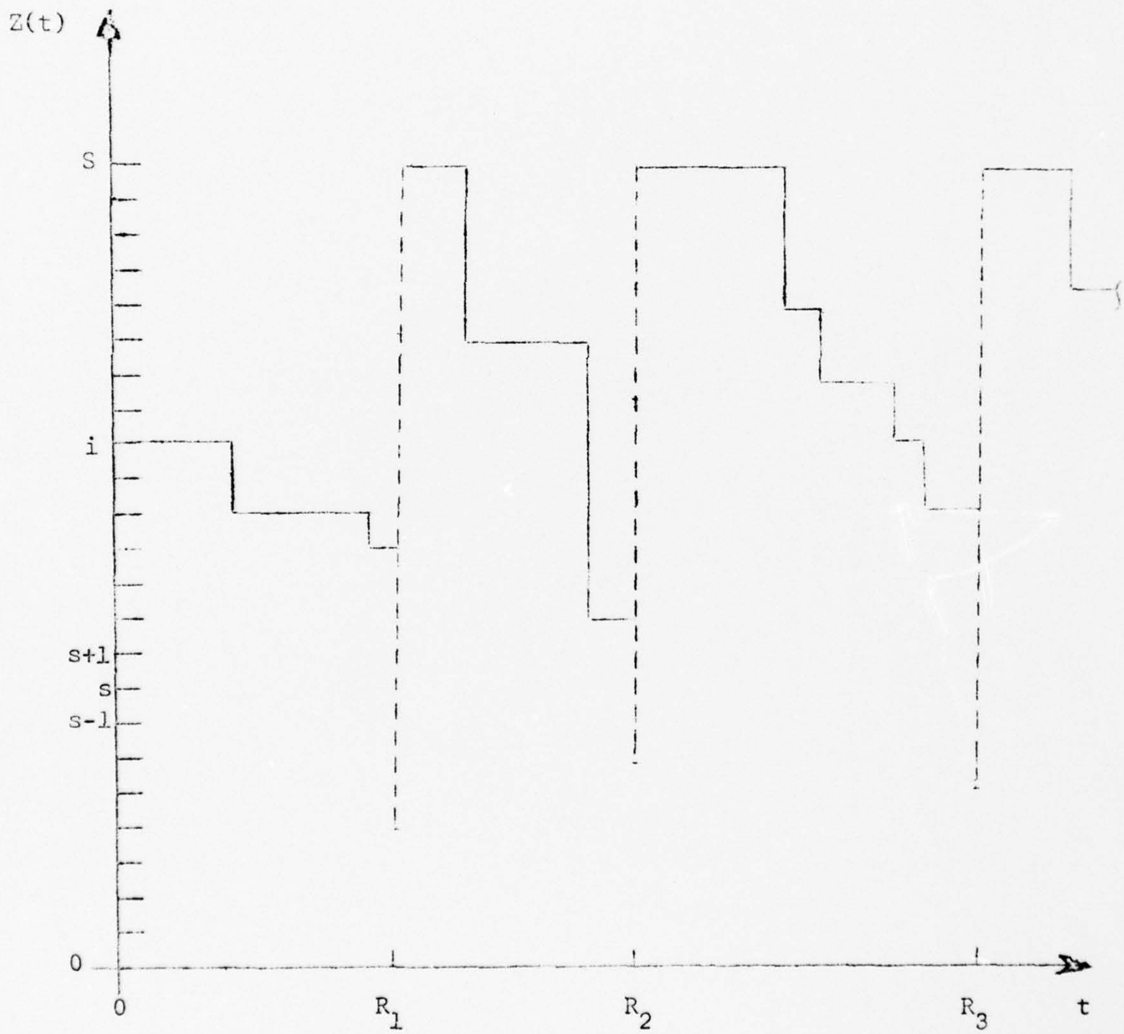


Figure 3.1: A sample realization of the inventory position with the (s,S) policy and random order sizes.

There are two possible approaches to obtain the stationary distribution of $Z(t)$. One approach is based on renewal theory and the other uses the theory of semi-Markov process. Though the underlying arguments remain the same, we shall use the second approach in our analysis.

We restrict the use of the term 'limiting distribution' in the context of the stochastic processes with discrete index parameter such as Markov chains; whereas, the term 'stationary distribution' will be used in the context of continuous index parameter stochastic processes such as $\{Z(t), t \geq 0\}$.

The stochastic process $\{X(t), t \geq 0\}$ changes its state at demand arrival epochs (decreases), at completions of repairs (increases), and at arrivals of procurement orders (increases). The state space of this process is the set $Z = \{S, S-1, \ldots, 1, 0, -1, -2, \ldots\}$. The positive values of $X(t)$ indicate on-hand inventory while the negative values indicate the existence of backorders $B(t)$ at time t; that is,

$$B(t) = \max(0, -X(t)).$$

In order to obtain the stationary distribution of $X(t)$, we assume that the procurement lead time is a constant τ.
For $t \geq \tau$ let

$$x_{ij}(t) = \Pr\{X(t) = j \mid Z(0) = i\} \qquad i \in E, \quad j \in Z \tag{3.4}$$

Anything on order from external supplier at time $t - \tau$ will have arrived by time t and anything ordered after time $t - \tau$ will arrive after time t. Thus the number of units received via

procurement during $(t-\tau, t]$ is $O(t-\tau)$. The number of units arriving from the repair shop during $(t-\tau, t]$ is $C(t-\tau, t)$. Therefore,

$$X(t) = X(t-\tau) + O(t-\tau) + C(t-\tau, t) - D_r(t-\tau, t) - D_c(t-\tau, t).$$

But $C(t-\tau, t) = Q(t-\tau) + D_r(t-\tau, t) - Q(t)$; thus, we get

$$X(t) = Z(t-\tau) - Q(t) - D_c(t-\tau, t).$$

Then Eq. (3.4) can be written as

$$\begin{aligned}
(3.5)\quad x_{ij}(t) &= \sum_{k\varepsilon E} \Pr\{X(t) = j \mid Z(t-\tau) = k;\ Z(0) = i\} \cdot \Pr\{Z(t-\tau) = k \mid Z(0) = i\} \\
&= \sum_{k\varepsilon E} \Pr\{Z(t-\tau) - Q(t) - D_c(t-\tau, t) = j \mid Z(t-\tau) = k;\ Z(0) = i\} \\
&\qquad \cdot \Pr\{Z(t-\tau) = k \mid Z(0) = i\} \\
&= \sum_{k\varepsilon E} \Bigg[\sum_{m=0}^{\infty} \Pr\{Q(t) + D_c(t-\tau, t) = k-j \mid Q(t-\tau) = m;\ Z(t-\tau) = k; \\
&\qquad\qquad Z(0) = i\} \\
&\qquad \cdot \Pr\{Q(t-\tau) = m \mid Z(t-\tau) = k;\ Z(0) = i\} \Bigg] \\
&\qquad \cdot \Pr\{Z(t-\tau) = k \mid Z(0) = i\}.
\end{aligned}$$

The stationary distribution $x(j) = \lim_{t\to\infty} x_{ij}(t)$ is obtained by taking the limit (as $t\to\infty$) of the right hand side of Eq. (3.5). The stationary distributions for on-hand inventory and the backorders can easily be obtained knowing $x(j)$. The stationary distribution of $Q(t)$ will be derived during the process of obtaining $x(j)$.

In the next section, we assume the demand processes $\{D_c(t), t \geq 0\}$ and $\{D_r(t), t \geq 0\}$ are independent. In Section 3.3, these demand processes are assumed to be dependent. In Section 3.4 results are obtained for the case where the demand arrival process is Poisson. In Section 3.5, an (s,nQ) procurement policy is studied. In Section 3.6 results for the two special cases of complete recoverability and complete non-recoverability are outlined. In Section 3.7 some long-run averages are derived.

3.2 Independent Demand Processes

3.2.1 The Model

The case of independent demand processes arises when there are two independent streams of failure processes responsible for the recoverable and the non-recoverable demands.

The recoverable demands arrive at the supply point at the epochs of time $T_r^0 = 0$, T_r^1, $T_r^2, \ldots$, where the inter-arrival times $T_r^n - T_r^{n-1}$ $(n = 1,2,\ldots)$ are independent and positive random variables with common distribution function $A_r(t) = Pr\{T_r^n - T_r^{n-1} \leq t\}$, $(t \geq 0; n = 1,2,\ldots)$. Let ξ_r^n be the number of the units (repairable) turned in for replacement at n^{th} epoch. The order sizes $\xi_r^1, \xi_r^2, \ldots$ are independent, positive and integer-valued random variables with common probability distribution $\phi_r(j) = Pr\{\xi_r^n = j\}$, $(j = 1,2,\ldots;\ n=1,2,\ldots)$. These random variables are also independent of the arrival process $\{T_r^n\}$. It is assumed that the inter-arrival times and order sizes have finite means; that is,

$$a_r = \int_0^\infty tA_r(dt) < \infty \quad \text{and} \quad d_r^m = \sum_{j=1}^\infty j\phi_r(j) < \infty.$$

Similarly, the non-recoverable demands arrive at the epochs $T_c^0 = 0, T_c^1, T_c^2, \ldots$ and the inter-arrival times $T_c^n - T_c^{n-1}$ $(n = 1,2,\ldots)$ are independent and positive random variables with common distribution function $A_c(t) = \Pr\{T_c^n - T_c^{n-1} \leq t\}$, $(t \geq 0;\ n = 1,2,\ldots)$. The respective number of the units $\xi_c^1, \xi_c^2, \ldots$ demanded are independent, positive and integer-valued random variables with the common probability distribution $\phi_c(j) = \Pr\{\xi_c^n = j\}$, $(j = 1,2,\ldots;\ n=1,2,\ldots)$, and these are also independent of the arrival process $\{T_c^n\}$. It is assumed that $\phi_c(1) > 0$, and

$$a_c = \int_0^\infty tA_c(dt) < \infty \quad \text{and} \quad d_c^m = \sum_{j=1}^\infty j\phi_c(j) < \infty$$

The demand processes $\{D_r(t), t \geq 0\}$ and $\{D_c(t), t \geq 0\}$ are independent since $\{T_r^n\}$, $\{T_c^n\}$, $\{\xi_r^n\}$ and $\{\xi_c^n\}$ are independent. We also assume that A_r and A_c are non-arithmetic [23].

For any $n = 1,2,\ldots,$ let

$$A_c^{(n)}(t) = 0 \qquad \text{for } t < 0,$$

$$A_c^{(n)}(t) = \int_0^t A_c(t-y)A_c^{(n-1)}(dy) \qquad \text{for } t \geq 0,$$

and

$$\phi_c^{(n)}(j) = \sum_{k=0}^{j} \phi_c(j-k)\phi_c^{(n-1)}(k) \qquad \text{for } j = 0,1,\ldots,$$

where $A_c^{(0)}(t) = 1$ for $t \geq 0$, $A_c^{(0)}(t) = 0$ for $t < 0$, $\phi_c^{(0)}(0) = 1$ and $\phi_c^{(n)}(j) = 0$ for $n < j$. We shall use similar notation for $A_r(\cdot)$ and $\phi_r(\cdot)$.

Also, for $t \geq 0$ let

$N_c(t)$ = the number of non-recoverable requisitions arriving during the interval $(0,t]$,

$U(t)$ = the length of time interval between time t and the epoch of the first non-recoverable requisition arriving after t.

$v_k^c(t) = \Pr\{D_c(t) = k\}$, $k = 0,1,\ldots,$

$w_k^c(t, t+\tau) = \Pr\{D_c(t, t+\tau) = k\}$ $k = 0,1,\ldots,$ $\tau \geq 0,$

and

$w_k^{c*}(\tau) = \lim_{t\to\infty} w_k^c(t, t+\tau).$

Then

$$v_k^c(t) = \sum_{n=0}^{k} \Pr\{D_c(t) = k | N_c(t) = n\} \cdot \Pr\{N_c(t) = n\}$$

$$= \sum_{n=0}^{k} \phi_c^{(n)}(k) \cdot \{A_c^{(n)}(t) - A_c^{(n+1)}(t)\}.$$

As shown in the reference [23], we note that $v_k(t)$ satisfies the following integral equations,

(3.6) $$v_0^c(t) = 1 - A_c(t), \quad t \geq 0$$

and

(3.7) $$v_k^c(t) = \sum_{j=1}^{k} \int_0^t \phi_c(j) v_{k-j}^c(t - x) A_c(dx), \quad k = 1,2,\ldots .$$

Since A_c is non-arithmetic, we have

(3.8) $$\lim_{t\to\infty} \Pr\{U(t) \leq u\} = \frac{1}{a_c}\int_0^u \{1 - A_c(x)\}dx$$ (see p. 97, [23]).

For each $t \geq 0$, we have

(3.9) $$w_0^c(t,t+\tau) = \Pr\{U(t) > \tau\}, \text{ and}$$

(3.10) $$w_k^c(t,t+\tau) = \sum_{j=1}^{k}\int_0^{\tau} \phi_c(j)v_{k-j}^c(\tau - u)\cdot d\Pr\{U(t) \leq u\}$$

$$k = 1,2,\ldots .$$

From Eqs. (3.8 - 3.10) we get the stationary distribution

(3.11) $$w_k^{c*}(\tau) = \begin{cases} \frac{1}{a_c}\int_{\tau}^{\infty} \{1 - A_c(u)\}du & k = 0 \\ \frac{1}{a_c}\sum_{j=1}^{k}\int_0^{\tau}\phi_c(j)v_{k-j}^c(\tau - u)\cdot\{1 - A_c(u)\}du & \end{cases}$$

$$k = 1,2,\ldots .$$

Similar results can be obtained for the process $\{D_r(t), t \geq 0\}$.

3.2.2 The Stationary Distribution of the Process $\{Z(t), t \geq 0\}$

From the definition of the inventory position (Eq. (3.1)), it is clear that the stochastic process $\{Z(t), t \geq 0\}$ changes its state only at the arrival epochs $T_c^1, T_c^2, \ldots$. It remains unchanged at the epochs $T_r^1, T_r^2, \ldots$ because in the case of recoverable demands, $X(t)$ decreases and $Q(t)$ increases by the same amount with no change in $Z(t)$. Completions of repairs or arrivals of supplies from an external supplier do not result in a change in $Z(t)$.

It is clear that

$$Z(t) = Z(T_c^n) \quad \text{for} \quad T_c^n \leq t < T_c^{n+1};\ n = 0,1,\ldots,t \geq 0.$$

Moreover, we have

$$\Pr\{Z(T_c^{n+1}) = j;\ T_c^{n+1} - T_c^n \leq t \mid Z(T_c^0),\ Z(T_c^1),\ldots,Z(T_c^n);$$
$$T_c^0,\ T_c^1,\ \ldots,\ T_c^n\}$$

$$= \Pr\{Z(T_c^{n+1}) = j;\ T_c^{n+1} - T_c^n \leq t \mid Z(T_c^n);\ T_c^n\} \quad \text{almost surely,}$$

$$\text{for} \quad n = 0,1,\ldots,t \geq 0; \quad \text{and} \quad j \in E.$$

Thus $\{Z(t),\ t \geq 0\}$ is a semi-Markov process whose kernel is given by

$$Q(i,j;\ t) = \Pr\{Z(T_c^{n+1}) = j;\ T_c^{n+1} - T_c^n \leq t \mid Z(T_c^n) = i\} \quad i,j \in E.$$

Define

$$P(i,j) = \lim_{t\to\infty} Q(i,j;\ t)$$

Then $P(i,j) \geq 0$ and $\sum_{j\in E} P(i,j) = 1$, so that the $P(i,j)$ are the transition probabilities of the imbedded Markov chain $\{Z(T_c^n)\}$. To derive these results we have

$$Q(i,j;\ t) = 0 \qquad s + 1 \leq i \leq j \leq S - 1;$$
$$Q(i,j;\ t) = A_c(t)\phi_c(i - j) \qquad j < i,\ s + 2 \leq i \leq S;$$
$$Q(i,S;\ t) = A_c(t) \sum_{k\geq i-s} \phi_c(k) \qquad s + 1 \leq i \leq S.$$

Using the fact that $\lim_{t\to\infty} A_c(t) = 1$, we obtain

$$P(i,j) = 0 \qquad s + 1 \le i \le j \le S - 1,$$

$$P(i,j) = \phi_c(i-j) \qquad j < i;\ s + 2 \le i \le S,$$

$$P(i,S) = \sum_{k\ge i-s} \phi_c(k) \qquad s+1 \le i \le S.$$

Now we have the following:

$$P^n(i,j) = \Pr\{Z(T_c^n) = j \mid Z(T_c^0) = i\},\quad n = 0,1,\ldots;\ i,j \in E.$$

Let $Z_n = Z(T_c^n)$, $n = 0,1,\ldots$; then $\{Z_n\}$ is a Markov chain imbedded in the semi-Markov process $\{Z(t), t \ge 0\}$. The transition probability matrix of this chain is P. We first obtain the limiting distribution of the chain $\{Z_n\}$ and then the stationary distribution of the process $\{Z(t), t \ge 0\}$

Theorem 3.1. The limiting distribution $\nu(j) = \lim_{n\to\infty} P^n(i,j)$ for $i,j \in E$, of the imbedded Markov chain $\{Z_n\}$ exists and is given by

$$\nu(j) = \frac{m(S-j)}{1+M(S-s-1)},\quad j = s + 1,\ldots,S - 1, \tag{3.12}$$

$$\nu(S) = \frac{1}{1+M(S-s-1)}$$

where

$$M(k) = \sum_{\ell=1}^{k} m(\ell), \tag{3.13}$$

and $m(k)$ satisfies

$$(3.14)\qquad m(1) = \phi_c(1);\ m(k) = \phi_c(k) + \sum_{q=1}^{k-1} \phi_c(k-q)m(q),\ k = 2,3,\ldots .$$

Proof. From our assumption that $\phi_c(1) > 0$, it follows that the chain $\{Z_n\}$ is irreducible. From the theory of finite Markov chains [8], we know that in a finite irreducible Markov chain all states are positive recurrent. We consider the following two cases.

(i) $0 < \phi_c(1) < 1$.

In this case the chain is aperiodic. We know that for an irreducible, positive recurrent and aperiodic chain, the limiting distribution $\{\nu(j) > 0\}$ exists and is given by the equations

$$(3.15)\qquad \nu(j) = \sum_{i\varepsilon E} \nu(i)P(i,j), \qquad j \varepsilon E$$

and

$$(3.16)\qquad \sum_{j\varepsilon E} \nu(j) = 1.$$

Using Eq. (3.14) we can reduce Eq. (3.15) successively for $j = S-1, S-2,\ldots, s+1$ to

$$\nu(j) = m(S-j)\nu(S).$$

The normalizing condition given by Eq. (3.16) and the Eq. (3.13) lead to the desired results (Eq. (3.12)).

(ii) $\phi_c(1) = 1$

In this case the chain is periodic with period $S - s$. Also, the matrix $P(i,j)$ is doubly stochastic. We know that the limiting distribution for this chain exists and is given by the Eqs. (3.15) and (3.16) which on solving yield

$$\nu(j) = \frac{1}{S-s} \quad \text{for all } j \in E. \tag{3.17}$$

For $\phi_c(1) = 1$, from Eqs. (3.14) and (3.13) we have $m(k) = 1$ for all $k \in E$, and $M(S-s-1) = S-s-1$. Substituting these into Eq. (3.12) we get the relations given by Eq. (3.17).

Q.E.D.

We now find the stationary distribution of the process $\{Z(t), t \geq 0\}$.

Theorem 3.2. *For the process* $\{Z(t), t \geq 0\}$, *the stationary distribution* $\Pi(j) = \lim_{t\to\infty} \Pr\{Z(t) = j | Z(0) = i\}$; $i, j \in E$ exists and is given by

$$\Pi(j) = \nu(j) \quad j \in E \tag{3.18}$$

where $\nu(j)$ are given by Eq. (3.12).

Proof: According to our assumption, the probability that a transition will take place within an amount of time t, given that process has just entered state i and will next enter j is independent of i and j and is $A_c(t)$. Let $h(k)$ be the expected amount of time spent in a state k during each visit, then

$$h(k) = \sum_{j \in E} P(k,j) \int_0^\infty t A_c(dt)$$

$$= \sum_{j \in E} P(k,j)a_c.$$

$$= a_c < \infty \quad \text{for all} \quad k \in E.$$

For the case $0 < \phi_c(1) < 1$, when the chain is aperiodic we know that $\{\Pi(j) > 0\}$ exist and is given by

$$\Pi(j) = \frac{\nu(j)h(j)}{\sum_{k \in E} \nu(k)h(k)}$$

$$= \nu(j) .$$

For the case $\phi_c(1) = 1$, when the chain is periodic with period $S - s$, the mean recurrence time of state k is given by (p. 90, [14]),

$$h(k,k) = (S-s) \int_0^\infty tA_c(dt)$$

$$= (S-s)a_c \qquad \text{for all} \quad k \in E;$$

and

$$\Pi(j) = \frac{h(j)}{h(j,j)} = \frac{1}{S-s} .$$

Q.E.D.

We note that $\Pi(j)$ does not depend on the recoverable demand arrival process $\{A_r\}$ or order size distribution $\phi_r(\cdot)$.

3.2.3. The Stationary Distribution of the Process $\{X(t), t \geq 0\}$

To obtain the transient distribution of the process $X(t)$ given by Eq. (3.5), we first prove the following theorem.

Theorem 3.3. For independent demand processes, $Q(t)$ and $Z(t)$ are independent for any $t > 0$.

Proof: From Eq. (3.1) we have

$$Q(t) = Q(0) + D_r(t) - C(t) \quad \text{and} \quad Z(t) = Z(0) + OQ(t) - D_c(t)$$

As mentioned in Section 3.2.2, recoverable demands do not affect the inventory position and thus do not influence the procurement orders. Therefore $OQ(t)$ is independent of $D_r(t)$. Similarly, $C(t)$ is independent of $D_c(t)$. Since $D_r(t)$ and $D_c(t)$ are independent, we can write

$$\begin{aligned} &\Pr\{Q(t) = q(t) \mid Z(t) = z(t)\} \\ &= \Pr\{Q(0) + D_r(t) - C(t) = q(t) \mid Z(0) + OQ(t) - D^c(t) = z(t)\} \\ &= \Pr\{Q(0) + D_r(t) - C(t) = q(t)\} \end{aligned}$$

Thus $Q(t)$ and $Z(t)$ are independent.

Q.E.D.

Applying Theorem 3.3 and using the fact that $D_c(t - \tau, t)$, $Q(t)$ and $O(t - \tau) + X(t - \tau)$ are independent, we can rewrite the conditional transient distribution of $X(t)$ as follows. For $t \geq \tau$, we have

$$(3.19) \quad \Pr\{X(t) = j \mid Z(t - \tau) = k, \ Z(0) = i\}$$

$$= \sum_{m=0}^{\infty} \Pr\{D_c(t-\tau, t) + Q(t) = k-j \mid Q(t - \tau) = m; \ Q(t - \tau) + X(t - \tau) = k-m; \ Z(0) = i\} \cdot \Pr\{Q(t - \tau) = m \mid Z(0) = i\}$$

$$= \sum_{m=0}^{\infty} \Pr\{D_c(t-\tau,\ t) + Q(t) = k-j \mid Q(t-\tau) = m;\ Z(0) = i\}$$

$$\cdot \Pr\{Q(t-\tau,\ t) = m \mid Z(0) = i\}$$

$$= \Pr\{D_c(t-\tau,\ t) + Q(t) = k-j \mid Z(0) = i\}$$

$$= \sum_{d_c=0}^{k-j} \Pr\{Q(t) = k-j-d_c \mid Z(0) = i\} \cdot \Pr\{D_c(t-\tau,\ t) = d_c\},$$

since $D_c(t-\tau,\ t)$ does not depend on $Z(0)$.

Now following the theorem of convergence of independent processes [2], from Eqs. (3.5) and (3.19) we have

$$(3.20)\quad x(j) = \lim_{t\to\infty} \Pr\{X(t) = j \mid Z(0) = i\} \quad \text{for } j \in Z$$

$$= \sum_{k\in E} \left[\sum_{d_c=0}^{k-j} \lim_{t\to\infty} \Pr\{Q(t) = k-j-d_c \mid Z(0) = i\} \cdot \lim_{t\to\infty} \Pr\{D_c(t-\tau,\ t) = d_c\} \right] \cdot \Pi(k)$$

The above expression describes the stationary distribution of $X(t)$. Eq. (3.20) can be evaluated after obtaining the stationary distribution of $Q(t)$ and from Eqs. (3.11) and (3.18).

The stationary distribution of $Q(t)$ is obtainable using queueing theoretic methods. As indicated in [6], explicit analytical results are not available for a GI/G/C (general inter-arrival and repair time distributions and a finite number of repair facilities) system with bulk (batch) input. We shall consider the following special case:

GI/R/∞ (general arrival process, constant repair time = R, infinite repair facilities) system with batch input.

For any $t \geq R$, the number of units in repair at time t is equal to the number of the recoverable units that arrived in interval $(t - R, t]$; that is,

$$\Pr\{Q(t) = k - j - d_c | Z(0) = i\} = \Pr\{D_r(t - R, t) = k - j - d_c\}.$$

Following the steps similar to those used in deriving $\lim_{t\to\infty} w_k^c(t, t+\tau)$, we have

$$(3.21)\quad \lim_{t\to\infty} \Pr\{Q(t) = k' | Z(0) = i\}$$

$$= \begin{cases} \dfrac{1}{a_r} \displaystyle\int_R^\infty \{1 - A_r(u)\}du & k' = 0 \\ \dfrac{1}{a_r} \displaystyle\sum_{j=1}^{k'} \int_0^R \phi_r(j')\, v_{k-j}^r(R - u).\, \{1 - A_r(u)\}du & k' = 1,2,\ldots \end{cases}$$

where $v_{k'}^r(t) = \sum_{n=0}^{k'} \phi_r^{(n)}(k')\{A_r^{(n)}(t) - A_r^{(n+1)}(t)\}.$

From Eqs. (3.11), (3.18) and (3.21), we can obtain $x(j)$.

3.3 Dependent Demand Processes

3.3.1 The Model

We view the case of dependent demand processes arising from a single failure process.

The demands arrive at the supply point at the epochs of time $T^0 = 0, T^1, T^2, \ldots$. The inter-arrival times $T^n - T^{n-1}$ $(n = 1,2,\ldots)$ are independent and positive random variables with common distribution

function $A(t) = Pr\{T^n - T^{n-1} \leq t\}$, $(t \geq 0; n = 1,2,\ldots)$. The respective number of units demanded are independent, positive and integer-valued random variables with the common probability distribution $\phi(\cdot)$. These are also independent of the arrival process. It is assumed that $\phi(1) > 0$; and

$$a = \int_0^\infty t\, A(t) < \infty \quad \text{and} \quad d^m = \sum_{j=1}^\infty j\, \phi(j) < \infty .$$

Here the demand process $\{D_r(t), t \geq 0\}$ and $\{D_c(t), t \geq 0\}$ are, in general, dependent. Similar to that in previous section, we shall denote the n-fold convolution of $\phi(\cdot)$ and $A(\cdot)$ by $\phi^{(n)}(\cdot)$ and $A^{(n)}(\cdot)$, respectively. For $t \geq 0$, let

$N(t)$ = the number of total requisitions arriving in $(0,t]$,

$U(t)$ = the length of the time interval between t and the epoch of the next arrival,

$$v_{k_1,k_2}(t) = Pr\{D_c(t) = k_1;\ D_r(t) = k_2\} \qquad k_1,k_2 = 0,1,\ldots ,$$

$$w_{k_1,k_2}(t,t+\tau) = Pr\{D_c(t,t+\tau) = k_1;\ D_r(t,t+\tau) = k_2\}$$
$$k_1,k_2 = 0,1,\ldots, t \geq 0 ,$$

$$w^*_{k_1,k_2}(\tau) = \lim_{t\to\infty} w_{k_1,k_2}(t,t+\tau) .$$

We shall derive these for the two inspection models discussed earlier.

Batch Model:

In batch model, where the entire batch of the failed units is either repaired or condemned, inspections are repeated independent

Bernoulli trials with probability $r_B (0 \leq r_B < 1)$ of sending a batch to the repair cycle and probability $(1 - r_B)$ of condemning the batch. This divides the requisitions into recoverable and non-recoverable types. For $t \geq 0$, let

$N_c(t)$ = the number of non-recoverable requisitions arriving in $(0,t]$, and

$N_r(t)$ = the number of recoverable requisitions in $(0,t]$.

We have the following.

$$Pr\{N_r(t) = k | N(t) = n\} = \binom{n}{k} r_B^k (1 - r_B)^{n-k}$$

$$k = 0,1,\ldots,n.$$

$$(3.22) \quad v_{k_1,k_2}(t) = \sum_{n=0}^{k_1+k_2} \left[\left\{ \sum_{k=0}^{k_2} \binom{n}{k} r_B^k (1 - r_B)^{n-k} \phi^{(k)}(k_2) \psi^{(n-k)}(k_1) \right\} \cdot [A^{(n)}(t) - A^{(n+1)}(t)] \right] \quad k_1, k_2 = 0,1,\ldots .$$

Following an approach similar to that used in deriving Eqs. (3.9) and (3.10), we get

$$(3.23) \quad w_{k_1,k_2}(t, t + \tau) = Pr\{U(t) > \tau\}, \quad k_1,k_2 = 0, \text{ and}$$

$$(3.24) \quad w_{k_1,k_2}(t,t+\tau) = r_B \cdot \sum_{j=1}^{k_2} \int_0^\tau \phi(j) v_{k_1,k_2-j}(\tau - u) dPr\{U(t) \leq u\}$$

$$+ (1 - r_B) \cdot \sum_{j=1}^{k_1} \int_0^\tau \phi(j) v_{k_1-j,k_2}(\tau - u) dPr\{U(t) \leq u\}$$

$$k_1, k_2 = 1,2,\ldots .$$

The stationary distribution is given by

$$(3.25)\quad v_{k_1,k_2}(\tau) = \begin{cases} \dfrac{1}{a}\displaystyle\int_{\tau}^{\infty}\{1 - A(u)\}du & k_1,k_2=0 \\ r_B \cdot \dfrac{1}{a}\displaystyle\sum_{j=1}^{k_2}\int_0^{\tau} \phi(j)v_{k_1,k_2-j}(\tau - u)\{1 - A(u)\}du & \\ +(1 - r_B)\cdot\dfrac{1}{a}\displaystyle\sum_{j=1}^{k_1}\int_0^{\tau} \phi(j)v_{k_1-j,k_2}(\tau - u)\{1 - A(u)\}du & \end{cases}$$

$$k_1,k_2=1,2,\ldots .$$

Unit Model: In the unit model, each unit in a batch is inspected to determine if it will be repaired or be scrapped. The inspections are repeated Bernoulli trials with probability r_U $(0 \le r_U < 1)$ of repairing a unit and probability $(1 - r_U)$ of scrapping the unit. This divides the units into recoverable and non-recoverable classes. We can write

$$(3.26)\quad v_{k_1,k_2}(t) = \binom{k_1+k_2}{k_1}(1 - r_U)^{k_1} r_U^{k_2} \left[\sum_{n=0}^{k_1+k_2} \phi^{(n)}(k_1+k_2)(A^{(n)}(t)-A^{(n+1)}(t))\right]$$

$$k_1,k_2 = 0,1,2,\ldots .$$

Also,

$$(3.27)\quad w_{k_1,k_2}(t,t + \tau) = \Pr\{U(t) > \tau\} \qquad k_1,k_2 = 0$$

$$(3.28)\quad w_{k_1,k_2}(t,t + \tau) = \sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\left\{\binom{j_1+j_2}{j_1}(1 - r_U)^{j_1}(r_U)^{j_2}\int_0^{\tau}\phi(j_1+j_2)\cdot v_{k_1-j,k_2-j_2}(\tau - u)d\Pr\{U(t) \le u\right\}$$

$$k_1,k_2 = 1,2,\ldots .$$

The stationary distribution is given by

$$(3.29)\quad w^*_{k_1,k_2}(\tau) = \begin{cases} \dfrac{1}{a}\displaystyle\int_\tau^\infty \{1 - A(u)\}du & k_1,k_2 = 0 \\ \dfrac{1}{a}\displaystyle\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\left\{\binom{j_1+j_2}{j_1}(1-r_u)^{j_1} r_u^{j_2}\right. \\ \quad \left.\cdot \displaystyle\int_0^\tau \phi(j_1+j_2) v_{k_1-j_1,\, k_2-j_2}(\tau-u)\{1-A(u)\}du\right\} \\ & k_1,k_2 = 1,2,\ldots \end{cases}$$

3.3.2 The Stationary Distribution of the Process $\{Z(t), t \geq 0\}$

When a single process generates failures, $\{Z(t), t \geq 0\}$ changes its state possibly only at arrival epochs $T^1, T^2, \ldots$. Completions of repairs or arrivals of procurement orders do not change $Z(t)$. Following the arguments similar to those presented in Section 3.2.2, we see that $\{Z(t), t \geq 0\}$ is a semi-Markov process whose kernel is given by

$$Q(i,j;\, t) = \Pr\{Z(T^{n+1}) = j;\ T^{n+1} - T^n \leq t \mid Z(T^n) = i\}$$
$$i,j \in E.$$

In the following two subsections we obtain the kernel and the stationary distribution of $\{Z(t), t \geq 0\}$ for the two models.

3.3.2.1 Batch Model

Upon a demand arrival, the following two outcomes are possible:

(i) the batch is condemned (prob. = $1 - r_B$), $X(t)$ decreases and $Z(t)$ changes, and

(ii) the batch is found repairable (prob. = r_B), $X(t)$ decreases and $Q(t)$ increases by the same amount, and, therefore, $Z(t)$ remain unchanged.

Then,

$$Q(i,j;\ t) = 0 \qquad s + 1 \le i < j \le S - 1;$$

$$Q(i,j;\ t) = A(t)r_B \qquad s + 1 \le j = i < S;$$

$$Q(i,j;\ t) = A(t)(1 - r_B)\phi(i - j) \qquad j < i, s + 2 \le i \le S;$$

$$Q(i,S;\ t) = A(t)(1 - r_B) \sum_{k \ge i-s} \phi(k) \qquad s + 1 \le i \le S - 1;$$

$$Q(S,S;\ t) = A(t)\{r_B + (1 - r_B) \sum_{k \ge S-s} \phi(k)\}.$$

Using the fact that $\lim_{t\to\infty} A(t) = 1$, we obtain $P(i,j) = \lim_{t\to\infty} Q(i,j;\ t)$. The transition probability matrix $P(i,j)$ of the imbedded Markov chain is shown on next page. We have the following theorem.

Theorem 3.4. The stationary distribution $\Pi(j) = \lim_{t\to\infty} \{Z(t) = j | Z(0) = i\}$, $i,j \in E$ exists and is given by Eq. (3.18).

Proof. From the matrix $P(i,j)$ it is clear that the chain is irreducible since $\phi(1) > 0$ and $0 \le r_B < 1$. Therefore all the states are positive recurrent. For $0 < \phi(1) < 1$, the chain is aperiodic and limiting distribution is given by Eqs. (3.15) and (3.16) which on solving yield Eq. (3.12). For $\phi(1) = 1$, the chain is aperiodic for $0 < r_B < 1$ and is periodic with period $S-s$ for $r_B=0$. Therefore the limiting distribution is given by Eq. (3.17).

Proceeding as in the proof of Theorem 3.2. we obtain the desired result.

Q.E.D.

i \ j	s + 1	s + 2		S - 2	S - 1	S
s + 1	r_B	0		0	0	$(1-r_B)\cdot \sum_{k\geq 1} \phi(k)=(1-r_B)$
s + 2	$(1-r_B)\cdot\phi(1)$	r_B		0	0	$(1-r_B)\cdot \sum_{k\geq 2} \phi(k)$
S - 2	$(1-r_B)\cdot\phi(S-s-3)$	$(1-r_B)\cdot\phi(S-s-4)$		r_B	0	$(1-r_B) \sum_{k\geq S-s-2} \phi(k)$
S - 1	$(1-r_B)\cdot\phi(S-s-2)$	$(1-r_B)\cdot\phi(S-s-3)$		$(1-r_B)\cdot\phi(1)$	r_B	$(1-r_B) \sum_{k\geq S-s-1} \phi(k)$
S	$(1-r_B)\cdot\phi(S-s-1)$	$(1-r_B)\cdot\phi(S-s-2)$		$(1-r_B)\cdot\phi(2)$	$(1-r_B)\cdot\phi(1)$	$r_B+(1-r_B) \sum_{k\geq S-s} \phi(k)$

3.3.2.2 Unit Model

Let

$$p(k) = \Pr\{\text{the number of non-recoverable units in a batch} = k\}$$

$$= \sum_{d=k}^{\infty} \Pr\{\text{the number of non-recoverable units in a batch} = k \mid \text{batch size} = d\}\Pr\{\text{batch size} = d\}.$$

$$(3.30) \qquad = \sum_{d=k}^{\infty} \binom{d}{k} r_U^{(d-k)} (1 - r_U)^k \cdot \phi(d).$$

Then upon the arrival of a demand, $Z(t)$ changes with probability $\sum_{k=1}^{\infty} p(k)$ and remains unchanged with probability $p(0) = 1 - \sum_{k=1}^{\infty} p(k)$. Moreover, for $0 < r_U < 1$, $0 < p(0) < 1$. We have

$$Q(i,j;\ t) = 0 \qquad s + 1 \le i < j \le S - 1;$$

$$Q(i,j;\ t) = A(t)p(0) \qquad s + 1 \le j = i < S;$$

$$Q(i,j;\ t) = A(t)p(i - j) \qquad j < i, s + 2 \le i < S;$$

$$Q(i,S;\ t) = A(t) \sum_{k \ge i-s} p(k) \qquad s + 1 \le i \le S - 1;$$

and

$$Q(S,S;\ t) = A(t) \cdot \{p(0) + \sum_{k \ge S-s} p(k)\}.$$

We can obtain the transition probability matrix of the imbedded Markov chain $Z_n = Z(T^n), n = 0,1,\ldots$. (noting that $\lim_{t\to\infty} A(t) = 1$).

Theorem 3.5. The stationary distribution $\Pi(j) = \lim_{t\to\infty} \Pr\{Z(t) = j \mid Z(0) = i\}$, $i,j \in E$ exists and is given by

$$(3.31) \qquad \Pi(j) = \frac{g(S-j)}{1+G(S-s-1)}, \quad j = s+1,\ldots,S-1;$$

$$\Pi(S) = \frac{1}{1+G(S-s-1)}$$

where

$$(3.32) \qquad G(k) = \sum_{\ell=1}^{k} g(\ell)$$

and $g(k)$ being determined recursively from the equations

$$(3.33) \qquad \begin{aligned} g(1) &= p(1)/(1 - p(0)) \\ g(k) &= [p(k) + \sum_{q-1}^{k-1} p(q)g(k-q)]/[1 - p(0)] \\ & \qquad\qquad k = 2,3,\ldots . \end{aligned}$$

Proof: Suppose $0 < r_U < 1$ then $0 < p(1) < 1$. Following arguments similar to those in the proof of Theorem 3.1, we note that the chain is irreducible and all the states are positive recurrent. Therefore the limiting distribution $\{\nu(j) > 0\}$ of the chain exists and is given by Eqs. (3.15) and (3.16). Using Eq. (3.33) we reduce Eq. (3.15) successively for $j = S-1, S-2, \ldots, s+1$ to

$$\nu(j) = g(S-j)\nu(S).$$

The normalizing condition (Eq. (3.16)) and the Eq. (3.32) yield

$$\nu(j) = \frac{g(S-j)}{1+G(S-s-i)} \quad j = s+1,\ldots,S-1.$$

$$\nu(S) = \frac{1}{1+G(S-s-1)} .$$

Now applying Theorem 3.2 we obtain the desired results.

Q.E.D.

The special cases of complete recoverability and complete non-recoverability are discussed in Section 3.6.

3.3.3 The Stationary Distribution of the Process $\{X(t), t \geq 0\}$

We consider a constant repair time R and an infinite number of repair facilities. It is assumed that $R \leq \tau$. In order to obtain the transient distribution $x_{ij}(t), t \geq \tau$, given by Eq. (3.5), we first obtain the joint probability distribution of $\{Q(t), t \geq 0\}$ and $\{Z(t), t \geq 0\}$. If the demand processes are independent then the two processes are independent (Theorem 3.3). This offered a considerable simplification in evaluating $x_{ij}(t)$ in Section 3.2.3.

Since only the repairable units received during $(t-R, t]$ are in the repair cycle at time t $(t \geq R)$, we can write,

$$(3.34)\quad \Pr\{Q(t) = m;\ Z(t) = k \mid Z(0) = i\}$$

$$= \sum_{z \in E} \Pr\{Q(t) = m;\ Z(t) = k \mid Z(t-R) = z;\ Z(0) = i\}$$

$$\cdot \Pr\{Z(t-R) = z \mid Z(0) = i\}$$

$$= \sum_{z \in E} \Pr\{D_r(t-R, t) = m;\ Z(t) = k \mid Z(t-R) = z;\ Z(0) = i\}$$

$$\cdot \Pr\{Z(t-R) = z \mid Z(0) = i\}, \quad m = 0, 1, \ldots, \text{ and } k \in E.$$

Let

$N^0(t-R, t)$ = the number of procurement orders placed during $(t-R, t]$,

and Y_i = the time after $t-R$ when the i^{th} order is placed, $i=1,2,\ldots$.

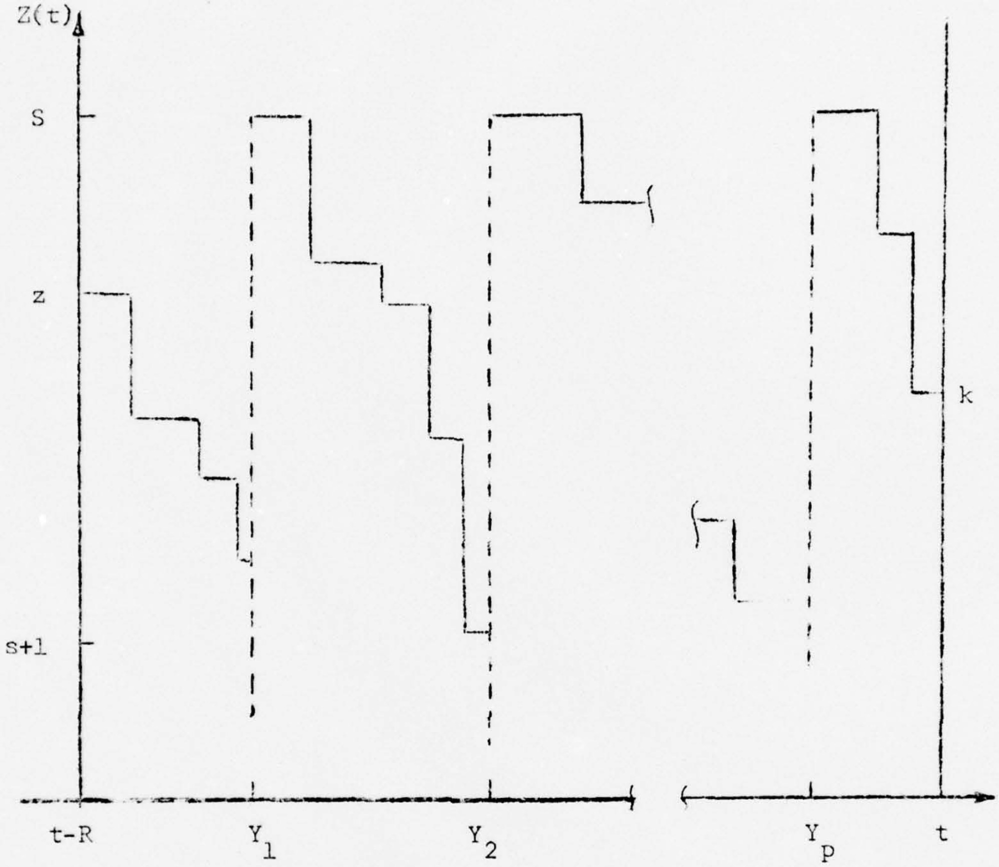


Figure 3.2: A sample realization of the process $\{Z(t),\ t\geq 0\}$ during the interval $(t-R,\ t]$.

The random variables $Y_1,\ Y_i - Y_{i-1}$ $(i=2,3,\ldots)$ are mutually independent. Furthermore, we have

(3.35) $$\Pr\{Y_1 \leq y \mid Z(t-R) = z\} = \sum_{k_1 \geq z-s} \Pr\{D_c(t-R,\ r-R+y) = k_1\};$$

and

$$\Pr\{Y_i - Y_{i-1} \leq y\} = \sum_{k_1 \geq S-s} \Pr\{D_c(y) = k_1\},\quad i=2,3,\ldots .$$

The process $\{Y_i\}$ is a renewal process when $z = S$, and it is a so-called delayed renewal process when $z \neq S$ (see Ross [14]). For $p = 1,2,\ldots$; let

$$F_{\ell,z}^{(p)}(t,y) = \Pr\{N^0(t-R,\ t) = p;\ Y_p \le y;\ D_r(t-R,\ t-R+Y_p) = \ell \mid Z(t-R) = z\}.$$

We have,

$$F_{\ell,z}^{(1)}(t,y) = \sum_{k_1 \ge z-s} \Pr\{D_c(t-R,\ t-R+y) = k_1;\ D_r(t-R,\ t-R+y) = \ell\}$$

and

$$(3.36)\quad F_{\ell,z}^{(p)}(t,y) = \int_0^y \sum_{k_2=0}^{\ell} F_{\ell-k_2,S}^{(p-1)}(t,\ y-u) F_{k_2,z}^{(1)}(t,du) \quad p = 2,3,\ldots\ .$$

For $0 \le y \le R$ and $p=1,2,\ldots$; we have,

$$(3.37)\quad \Pr\{Z(t) = k;\ Q(t) = m \mid N^0(t-R,\ t) = p;\ Y_p = y;\ D_r(t-R,\ t-R+y) = \ell;\ Z(t-R) = z\}$$
$$= \Pr\{D_c(t-R+y,\ t) = S-k;\ D_r(t-R+y,\ t) = m-\ell \mid N^0(t-R,\ t) = p;\ Y_p = y;\ D_r(t-R,\ t-R+y) = \ell;\ Z(t-R) = z\},$$

and for $p = 0$ we have

$$(3.38)\quad \Pr\{Z(t) = k;\ Q(t) = m \mid Z(t-R) = z\}$$
$$= \Pr\{D_c(t-R,\ t) = z-k;\ D_r(t-R,\ t) = m\}$$
$$= \begin{cases} w_{z-k,m}(t-R,t), & \text{for } z \ge k, \\ 0, & \text{otherwise}. \end{cases}$$

From Eqs. (3.36 - 3.38) we get

$$(3.39)\quad \Pr\{D_r(t-R,\ t) = m;\ Z(t) = k \mid Z(t-R) = z;\ Z(0) = i\}$$

$$= w_{z-k,m}(t-R,\ t) + \sum_{p=1}^{\infty} \sum_{\ell=0}^{m} \int_{y=0}^{R} w_{S-k,m-\ell}(t-R+y,\ t)\cdot F_{\ell,z}^{(p)}(t,dy).$$

Substituting Eq. (3.39) into Eq. (3.34) we obtain the joint probability distribution of $Q(t)$ and $Z(t)$, given by

$$(3.40)\quad \Pr\{Q(t) = m;\ Z(t) = k \mid Z(0) = i\}$$

$$= \sum_{z\varepsilon E} \left\{ w_{z-k,m}(t-R,\ t) + \sum_{p=1}^{\infty} \sum_{\ell=0}^{m} \int_{y=0}^{R} w_{S-k,m-\ell}(t-R+y,\ t) F_{\ell,z}^{(b)}(t,dy) \right\}$$

$$\cdot \Pr\{Z(t-R) = z \mid Z(0) = i\}.$$

We can now obtain the expressions for the two probability terms in Eq. (3.5).

Since $D_c(t-\tau,\ t)$ is independent of $Q(t-\tau)$ and $Z(t-\tau)$, and $Q(t)$ depends on $D_c(t-\tau,\ t)$ only through $D_r(t-R,\ t)$, we have

$$(3.41)\quad \Pr\{D_c(t-\tau,\ t) + Q(t) = k-j \mid Q(t-\tau) = m;\ Z(t-\tau) = k;\ Z(0) = i\}$$

$$= \sum_{d_c=0}^{k-j} \sum_{q=0}^{d_c} \Pr\{D_c(t-\tau,\ t-R) = q;\ D_c(t-R,\ t) = d_c-q;\ D_r(t-R,\ t) = k-j-d_c\}$$

$$= \sum_{d_c=0}^{k-j} \sum_{q=0}^{d_c} \left(\sum_{k_2=0}^{\infty} w_{q,k_2}(t-\tau,\ t-R) \right) \left(w_{d_c-q,k-j-d_c}(t-R,\ t) \right).$$

And

$$(3.42)\quad \Pr\{Q(t-\tau) = m \mid Z(t-\tau) = k;\ Z(0) = i\}$$

$$= \frac{\Pr\{Q(t-\tau)=m;\ Z(t-\tau)=k \mid Z(0) = i\}}{\Pr\{Z(t-\tau)=k \mid Z(0)=i\}}$$

The numerator of Eq. (3.42) can be obtained from Eq. (3.40). Substituting Eqs. (3.41) and (3.42) into Eq. (3.5) we get

$$(3.42)\quad x_{ij}(t) = \sum_{k\varepsilon E} \sum_{m=0}^{\infty} \left[\sum_{d=0}^{k-j} \sum_{q=0}^{d_c} \left[\sum_{k_2=0}^{\infty} w_{q,k_2}(t-\tau,\ t-R) \right] \right.$$

$$\left. \cdot\, w_{d_c-q,k-j-d_c}(t-R,\ t) \right] \cdot \left[\sum_{z\varepsilon E} \left\{ w_{z-k,m}(t-\tau-R,\ t-\tau) \right. \right.$$

$$+ \sum_{p=1}^{\infty} \sum_{\ell=0}^{m} \int_{y=0}^{R} w_{S-k,m-\ell}(t-\tau-R+y,\ t-\tau)$$

$$\left. \cdot\, F^{(p)}_{(\ell,z)}(t-\tau,\ dy) \right\}$$

$$\left. \cdot\, \Pr\{Z(t-\tau-R) = z \mid Z(0) = i\} \right]$$

The stationary distribution $x(j)$ can be obtained from Eq. (3.42) using the Laplace transform approach as suggested by Sivazlian [22].

3.4 The Case of Compound Poisson Demands

We now consider the case where the failure processes are Poisson processes. Both the cases of independent and dependent demand processes outlined in Sections 3.2 and 3.3 are investigated.

3.4.1 Independent Demand Processes

Suppose that non-recoverable and recoverable failures are generated by two independent Poisson processes with rates $\lambda_c(>0)$ and $\lambda_r(>0)$, respectively. Then, in terms of the model described in Section 3.2.1, $A_c(t) = 1 - e^{-\lambda_c t}$ and $A_r(t) = 1 - e^{-\lambda_r t}$, $t \geq 0$; $a_c = 1/\lambda_c$ $(< \infty)$ and $a_r = 1/\lambda_r (< \infty)$. Let $\phi_c(\cdot)$ $(\phi_c(1) > 0)$ and $\phi_r(\cdot)$ be the order size distributions of non-recoverable and re-coverable demands, respectively. Then $\{D_c(t), t \geq 0\}$ and $\{D_r(t), t \geq 0\}$ are independent compound Poisson processes with parameters λ_c and λ_r with compounding distributions $\phi_c(\cdot)$ and $\phi_r(\cdot)$, respectively. Consequently, from Eqs. (3.7) and (3.8) we have

$$(3.44) \qquad v_k^c(t) = \sum_{n=0}^{k} \phi_c^{(n)}(k) \frac{e^{-\lambda_c t}(\lambda_c t)^n}{n!}, \qquad k = 0,1,\ldots;$$

and

$$\lim_{t\to\infty} \Pr\{U(t) \leq u\} = \lambda_c \int_0^u e^{-\lambda_c x}\, dx = 1 - e^{-\lambda_c u}.$$

Thus from Eq. (3.10), $w_k^c(t, t+\tau) = v_k^c(\tau)$. Therefore

$$(3.45) \qquad w_k^{c*}(\tau) = v_k^c(\tau) \qquad \text{for } \tau \geq 0.$$

Since $A_c(t) = 1 - e^{-\lambda_c t}$, $t \geq 0$, and $\phi_c(\cdot)$ satisfy the conditions of Theorems 3.1 and 3.2, the stationary distribution $\Pi(j), j \in E$ is given by Eq. (3.18). To obtain the stationary distribution $x(j), j \in Z$, given by Eq. (3.20), we first obtain the stationary distribution of the process $\{Q(t), t \geq 0\}$.

Let G denote the repair time distribution of a unit and k be the number of repair facilities. Then using the nomenclature from queueing theory as given in Gross and Harris [6], the repair system is equivalent to a $M^{[\phi_r]}/G/k$ queueing system, and the stationary distribution of the process $\{Q(t), t \geq 0\}$ is equivalent to $\{p_n\}$, the stationary distribution of the number of customers in the queueing system. As mentioned by Gross and Harris, no analytical results for $\{p_n\}$ are available for the systems like $M^{[\phi_r]}/G/1$, $M^{[\phi_r]}/M/k$ and $M^{[\phi_r]}/M/1$. The results are available in the form of a generating function or a Laplace transform. In our following discussions, we shall consider a $M^{[\phi_r]}/G(T)/\infty$ repair system, where T is the mean (finite) of the repair time distribution. It is assumed that repair time is the same for all units in a batch and $Q(0) = 0$. Feeney and Sherbrooke [5] have shown that for such a system $\{p_n\}$ is a compound Poisson with parameter $\lambda_r T$ and compounding distribution $\phi_r(\cdot)$; that is,

$$(3.46) \qquad \lim_{t\to\infty} \Pr\{Q(t) = q\} = \sum_{n=0}^{q} \phi_r^{(n)}(q) \frac{e^{-\lambda_r T}(\lambda_r T)^n}{n!}$$

$$q = 0,1,\ldots .$$

We consider certain special cases.

(1) Geometric Order Size Distributions:

The geometric order size distribution has been extensively considered in inventory theory in the context of random order size, (for example, see Hadley and Whitin [7]). With geometric order sizes, the resulting demand processes $\{D_c(t), t \geq 0\}$ and $\{D_r(t), t \geq 0\}$ have stuttering Poisson distributions. As noted by Sherbrooke [18], a stuttering Poisson process offers considerable analytical simplification and is a 'natural' candidate for describing the demand processes in a number of practical situations. Let

$$\phi_c(0) = 0, \quad \phi_c(k) = (1 - \alpha_c)\alpha_c^{k-1}, \quad k = 1,2,\ldots .$$

and $$\phi_r(0) = 0, \quad \phi_r(k) = (1 - \alpha_r)\alpha_r^{k-1}, \quad k = 1,2,\ldots .$$

where $0 \leq \alpha_c < 1$ and $0 \leq \alpha_r < 1$.

From Eqs. (3.14) and (3.13) we have

$$m(1) = 1 - \alpha_c$$
$$m(2) = (1 - \alpha_c)\alpha_c + (1 - \alpha_c)(1 - \alpha_c) = 1 - \alpha_c$$
$$m(3) = (1 - \alpha_c)\alpha_c^2 + (1 - \alpha_c)\alpha_c(1 - \alpha_c) + (1 - \alpha_c)(1 - \alpha_c)$$
$$= 1 - \alpha_c .$$

It can be easily shown that

$$m(k) = 1 - \alpha_c, \quad \text{for } k \geq 1.$$

Then $M(S-s-1) = (S-s-1)(1-\alpha_c)$. Substituting these into Eq. (3.18) we get

(3.47) $$\Pi(j) = \frac{1-\alpha_c}{1+(S-s-1)(1-\alpha_c)} \qquad j = s+1, s+2, \ldots, S-1;$$

and $$\Pi(S) = \frac{1}{1+(S-s-1)(1-\alpha_c)} .$$

The distribution given by Eq. (3.45) is a stuttering Poisson distribution and is given by

(3.48) $$w^{c*}_{d_c}(\tau) = \begin{cases} e^{-\lambda_c \tau} & d_c = 0 \\ \alpha_c^{d_c} e^{-\lambda_c \tau} \sum_{j=1}^{d_c} \frac{1}{j!} \binom{d_c - 1}{j - 1} \left[\left(\frac{1-\alpha_c}{\alpha_c}\right) \lambda_c \tau\right]^j & d_c \geq 1. \end{cases}$$

For a $M^{[\phi_r]}/G(T)/\infty$ repair system, it follows from Eq. (3.46) that $\lim_{t\to\infty} \Pr\{Q(t) = q\}$ also has a stuttering Poisson distribution given by

(3.49) $$\lim_{t\to\infty} \Pr\{Q(t) = k-j-d_c \mid Q(0) = 0\}$$

$$= \begin{cases} e^{-\lambda_r T} & d_c = k - j \\ \alpha_r^{k-j-d_c} e^{-\lambda_r T} \sum_{n=1}^{k-j-d_c} \frac{1}{n!} \binom{k - j - d_c}{n - 1} \left[\frac{1-\alpha_r}{\alpha_r} \cdot \lambda_r T\right]^n & d_c < k - j. \end{cases}$$

Substituting Eqs. (3.47 - 3.49) into Eq. (3.20) we can obtain the stationary distribution $x(j)$, $j \in Z$.

(2) Unit Order Size:

When units are demanded one at a time, that is, when $\phi_c(1) = 1$ and $\phi_r(1) = 1$, it follows from Eqs. (3.18) and (3.45) that

$$\Pi(j) = \frac{1}{S-s} \qquad j \in E$$

and
$$w^{c*}_{d_c}(\tau) = \frac{e^{-\lambda_c \tau}(\lambda_c \tau)^{d_c}}{d_c!}, \qquad d_c \geq 0.$$

For a $M/G(T)/\infty$ repair system, from Eq. (3.46) it follows that

$$\lim_{t\to\infty} \Pr\{Q(t) = k - j - d_c\} = \frac{e^{-\lambda_r T}(\lambda_r T)^{k-j-d_c}}{(k - j - d_c)!}, \qquad k \geq j + d_c .$$

Substituting the above results into Eq. (3.20) we get

$$\text{(3.50)} \quad x(j) = \frac{1}{S-s} \sum_{k=s+1}^{S} \sum_{d_c=0}^{k-j} \frac{e^{-\lambda_c \tau}(\lambda_c \tau)^{d_c}}{d_c!} \frac{e^{-\lambda_r T}(\lambda_r T)^{k-j-d_c}}{(k - j - d_c)!}$$
$$= \frac{1}{S-s} \sum_{k=s+1}^{S} \frac{e^{-(\lambda_c \tau+\lambda_r T)}(\lambda_c \tau+\lambda_r T)^{k-j}}{(k - j)!}$$

As noted by Richards [13], it is interesting to compare the results given by Eq. (3.50) with the results given by Hadley and Whitin ([7], pp. 183-187) for a continuous review consumable item inventory system with Poisson demands with parameter λ and a constant procurement lead time τ, where

$$\text{(3.51)} \qquad x(j) = \frac{1}{S-s} \sum_{k=s+1}^{S} \frac{e^{-\lambda\tau}(\lambda\tau)^{k-j}}{(k-j)!} .$$

Thus a simple change of parameter from $\lambda\tau$ to $(\lambda_c \tau + \lambda_r T)$ makes Eqs. (3.51) and (3.50) identical.

We conclude by emphasizing that results obtained here hold for the case where an item has n possible independent failure modes. For mode i, the failure process is a Poisson process with rate λ_i and the order size distribution is $\phi_i(\cdot)$. Furthermore, if the failure mode belongs to a set R, the units are recoverable; but if the mode belongs to

the complement set C, the units are non-recoverable. For this case, the results obtained in this section hold with $\lambda_r = \sum_{i\varepsilon R} \lambda_i$, $\lambda_c = \sum_{i\varepsilon C} \lambda_i$, $\phi_r(\cdot) = \frac{1}{\lambda_r} \sum_{i\varepsilon R} \lambda_i\phi_i(\cdot)$ and $\phi_c(\cdot) = \frac{1}{\lambda_r} \sum_{i\varepsilon C} \lambda_i\phi_i(\cdot)$.

3.4.2 Dependent Demand Processes

Suppose the failures are generated by a Poisson process with rate λ (>0); that is, $A(t) = 1 - e^{-\lambda t}$, $t \geq 0$. Also, let $\phi(\cdot)$ denote the order size distribution. We examine this case for batch and unit inspection models.

Batch Model:

Let r_B be the probability that a batch of failed unit is recoverable, then from Eq. (3.22) we have

$$(3.52)\quad \Pr\{D_r(t) = k_1;\ D_c(t) = k_2\}$$

$$= \sum_{n=0}^{k_1+k_2} \sum_{k=\max(0,n-k_1)}^{k_2} \binom{n}{k} r_B^k(1 - r_B)^{n-k}\phi^{(k)}(k_2)\phi^{(n-k)}(k_1)\, \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= \sum_{k=0}^{k_2} \frac{e^{-\lambda r_B t}(\lambda r_B t)^k}{k!}\, \phi^{(k)}(k_2) \sum_{n-k=0}^{k_1} \frac{e^{-\lambda(1-r_B)t}(\lambda(1-r_B)t)^{n-k}}{(n-k)!}\, \phi^{(n-k)}(k_1)$$

From (3.52) it follows that the demand processes $\{D_r(t), t \geq 0\}$ and $\{D_c(t), t \geq 0\}$ are independent compound Poisson processes with parameters $r_B\lambda$ and $(1 - r_B)\lambda$, respectively, and have a common compounding distribution, $\phi(\cdot)$. Therefore, the results of Section 3.4.1 hold for the batch model with $\lambda_r = r_B\lambda$, $\lambda_c = (1 - r_B)\lambda$ and $\phi_r(\cdot) = \phi_c(\cdot) = \phi(\cdot)$.

Unit Model:

Let r_U be the probability that a unit in a batch is recoverable. Then from Eqs. (3.26) and (3.29) the demand distribution is given by

$$(3.53)\quad v_{k_1,k_2}(t) = Pr\{D_c(t) = k_1,\ D_r(t) = k_2\}$$
$$= \binom{k_1+k_2}{k_2}(1-r_U)^{k_1} r_U^{k_2} \left[\sum_{n=0}^{k_1+k_2} \phi^{(n)}(k_1+k_2) \frac{e^{-\lambda t}(\lambda t)^n}{n!}\right],$$

and

$$(3.54)\quad w^*_{k_1,k_2}(\tau) = v_{k_1,k_2}(\tau) \qquad \text{for } k_1,k_2 = 0,1,\ldots .$$

The stationary distributions $\Pi(j), j \in E$ and $x(j), j \in Z$ can be obtained from Eqs. (3.31) and (3.43), respectively. The derivation of $x(j)$, however, is computationally complex for a general $\phi(\cdot)$. We consider the special case of unit order size, that is, $\phi(1) = 1$. Using the fact that $\phi^{(n)}(k_1 + k_2) = 1$ for $n = k_1 + k_2$ and $\phi^{(n)}(k_1 + k_2) = 0$ for $n \neq k_1 + k_2$, it follows from Eq. (3.53) that

$$(3.55)\quad Pr\{D_c(t) = k_1;\ D_r(t) = k_2\}$$
$$= \binom{k_1+k_2}{k_1}(1-r_U)^{k_1} r_U^{k_2} \frac{e^{-\lambda t}(\lambda t)^{k_1+k_2}}{(k_1+k_2)!}$$
$$= \frac{e^{-r_U\lambda t}(r_U\lambda t)^{k_2}}{k_2!} \cdot \frac{e^{-(1-r_U)\lambda t}((1-r_U)\lambda t)^{k_1}}{k!} .$$

Thus $D_c(t)$ and $D_r(t)$ are independent Poisson processes with parameters $(1 - r_U)\lambda$ and $r_U\lambda$, respectively. Therefore, the results for the case of unit order size obtained in Section 3.4.1 hold in this

situation with $\lambda_r = r_U\lambda$, $\lambda_c = (1 - r_U)\lambda$, and $\phi_r(1) = \phi_c(1) = 1$. In addition, Eqs. (3.52) and (3.55) lead to the obvious conclusion that for the unit order size distribution, the batch and unit models are identical.

3.5 The Uniform Distribution of Inventory Position and the (s,nQ) Procurement Policy.

From the previous sections, we make the following observations on the stationary distribution of inventory position under an (s,S) procurement policy. For the case where the demand processes are independent, it is clear from Theorem 3.2 that $\Pi(j)$, $j \in E$, is uniform over $\{s + 1,\ldots,S\}$ for $\phi_c(1) = 1$, and is independent of $\phi_r(\cdot)$. Similarly, when the demand processes are dependent and $\phi(1) = 1$, Theorems 3.4 and 3.5 imply that $\Pi(j)$ is uniform over $\{s + 1,\ldots,S\}$. Furthermore, we can derive the conditions for which the uniform distribution is obtained. This is done in the following lemma.

Lemma 3.1: Under an (s,S) policy, $\Pi(j)$ is uniform over $\{s + 1,\ldots,S\}$ if and only if

(a) $\phi_c(1) = 1$ under the conditions of Section 3.2,

and (b) $\phi(1) = 1$ under the conditions of Section 3.3.

Proof: We have already discussed the 'if' part. For the 'only if' part, assume $\Pi(j)$, is uniform, that is,

$$\Pi(j) = \frac{1}{S-s} \quad \text{for } j \in E. \tag{3.56}$$

(a) From Eqs. (3.18), and (3.56) it follows that $m(s-j) = 1$, $s+1 \le j \le S-1$; and $M(S-s-1) = S-s-1$. Substituting into Eq. (3.14) we get $\phi_c(1) = 1$.

(b) From Theorem 3.4, it follows that above proof also holds for the batch model; that is, $\phi(1) = 1$. For the unit model, Eq. (3.56) implies that $g(S-j) = 1$ for $s + 1 \le j \le S - 1$ and $G(S-s-1) = 1$. Eq. (3.33) then implies that $p(1) = 1 - p(0)$ and $p(k) = 0$ for $k = 2,3,\ldots$. Substituting this into Eq. (3.30). we get $\phi(1) = 1$.

Q.E.D.

Because of its mathematical simplicity, the uniform distribution of inventory position has been extensively considered in inventory theory. For example see Hadley and Whitin ([7], pp. 181-183), Simon [19], and Sivazlian [22]. But Lemma 3.1 suggests that under an (s,S) policy, $\Pi(j)$ cannot be uniform unless (a) $\phi_c(1) = 1$ or (b) $\phi(1) = 1$. However, the uniform stationary distribution is recaptured under an (s,nQ) policy for arbitrary order size distributions. We first describe this policy.

<u>The (s,nQ) Policy:</u>

Under a continuous review policy, when the inventory position falls to the level s or below, nQ units are ordered where n is the largest integer such that the subsequent inventory position is between $s + 1$ and $s + Q$. Figure 3.3 shows a sample realization of inventory position under this policy. The state space of the process $\{Z(t), t \ge 0\}$ is the set $E' = \{i \mid s+1 \le i \le s+Q,\ i \text{ integers}\}$.

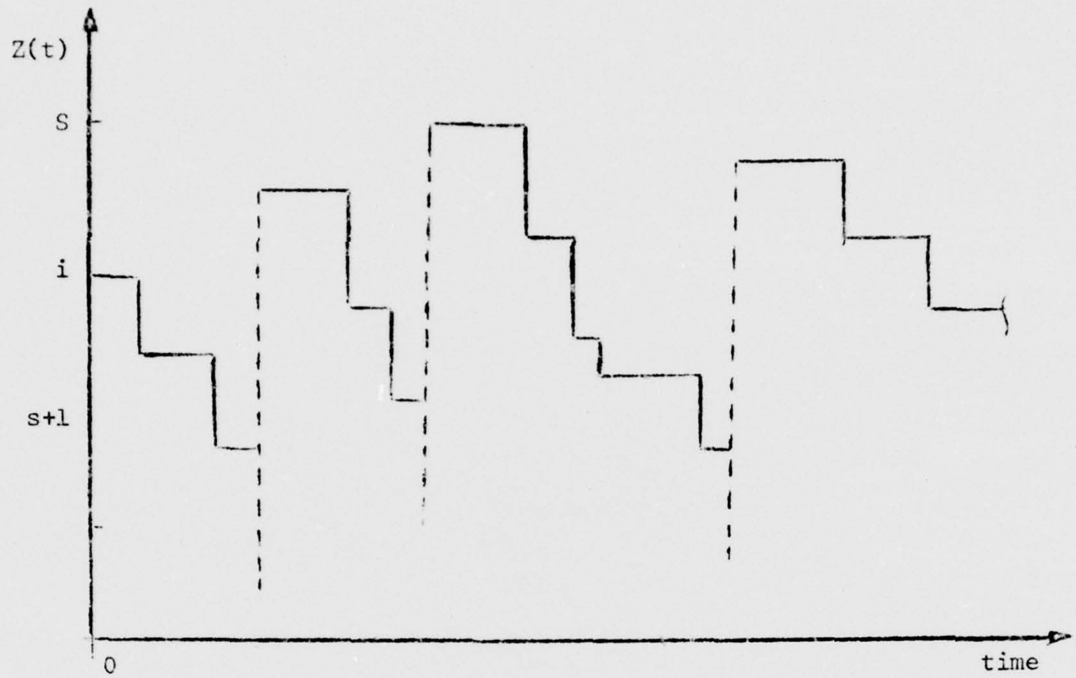


Figure 3.3: A sample realization of the inventory position with the (s,nQ) policy and random order sizes.

For a consumable item inventory system, it was shown by Simon [9] that the stationary distribution of inventory position (units on hand + units on order - number of backorders) exists and is uniform over $\{s+1,\ldots,s+Q\}$. In the following theorem, we extend his results for our version of a recoverable item inventory system.

Theorem 3.6. Under an (s,nQ) policy, $\Pi(j) = \frac{1}{Q}$, $j \in E'$ both for independent and dependent demand processes under the conditions of Sections 3.2 and 3.3, respectively.

Proof. We obtain the transition probabilities $P(i,j)$, $i, j \in E'$ of the imbedded Markov chain for the process $\{Z(t), t \geq 0\}$ under an (s,nQ) policy.

(a) Independent demand processes: Proceeding as in Section 3.2.1 for an (s,S) policy, we obtain

$$P(i,j) = \sum_{n=1}^{\infty} \phi_c(nQ - (j-i)), \qquad j \geq i;$$

and
$$P(i,j) = \sum_{n=0}^{\infty} \phi_c(nQ + (i-j)), \qquad j < i.$$

From the assumption that $\phi_c(1) > 0$, the Markov chain $\{Z_n\}$ is irreducible and its limiting distribution $\nu(j)$, $j \in E'$ is given by Eq. (3.12). Solving these equations we get $\nu(j) = \frac{1}{Q}$. By theorem 3.2, $\Pi(j) = \frac{1}{Q}$.

(b) Dependent demand processes: In this case, we shall consider both batch and unit models of inspection. For the batch model, proceeding as in Section 3.3.2.1, we have

$$P(i,j) = (1-r_B) \sum_{n=1}^{\infty} \phi(nQ + i - j), \qquad j > i;$$

$$P(i,j) = r_B + (1-r_B) \sum_{n=1}^{\infty} \phi(nQ), \qquad j = i;$$

and
$$P(i,j) = (1-r_B) \sum_{n=0}^{\infty} \phi(nQ + i - j), \qquad j < i.$$

The matrix P here is easily seen to be doubly stochastic , and the rest of the proof is similar to that in case (a) above.

Proceeding as in Section 3.3.2.2, we obtain the following for the unit model.

$$P(i,j) = \sum_{n=1}^{\infty} p(nQ + i - j) \qquad j > i$$

and

$$P(i,j) = \sum_{n=0}^{\infty} p(nQ + i - j) \qquad j \leq i.$$

From the assumption that $\phi(1) > 0$, it is clear that Eq. (3.30) implies $0 < p(1) < 1$ for $0 < r_U < 1$. The rest of the proof is similar to case (a). For $r_U = 0$, $p(k) = \phi(k)$ and the resulting situation is the same as case (a), with $\phi_c(\cdot) = \phi(\cdot)$ and $\phi_r(k) = 0$ for $k \geq 1$.

Q.E.D.

When the demand processes are independent, the stationary distribution of the process $\{X(t), t \geq 0\}$ under an (s,nQ) policy is the same as obtained in Section 3.2.3 for an (s,S) policy. Also, from the discussion in Section 3.4.2, it follows that this stationary distribution is the same under both policies when a Poisson process generates the failures and a batch model is used for inspection. Similar results hold for the stationary distribution of the process $\{Q(t), t \geq 0\}$. The derivation of these stationary distributions under an (s,nQ) policy for general dependent demand processes is not included in this study.

We emphasize that the (s,nQ) policy has received an appreciable acceptance in practice [19], because in addition to the mathematical advantage of a uniform distribution of inventory position, it permits the use of an economic lot size Q. The difference between an (s,S) and an (s,nQ) policy is that $Z(t)$, in an (s,nQ) policy is in E' immediately after placing an order, whereas in an (s,S) policy it is always S immediately after placing an order. The two policies are the same

if $\phi_c(1) = 1$ and $\phi(1) = 1$ for independent and dependent demand process, respectively.

3.6 Special Cases.

3.6.1 Complete Recoverability (no condemnations).

In this case, all failed units are recoverable with probability one. The system experiences only one type of demand (recoverable) and there is no distinction between independent and dependent demand processes. There are no procurement and inspection functions in the system and consequently, the external supplier and inspection facilities are eliminated from the list of entities in Figure 1.1. The resulting system, also referred to as a conservative system, resupplies itself from the repair facilities.

System demands can be considered to arise from a single arrival process $\{A_r(t), t \geq 0\}$ with order size distribution $\phi_r(\cdot)$ as described in Section 3.2. The inventory position $Z(t)$ remains constant for all $t \geq 0$. Let $Z(t) = S$, $t \geq 0$. The problem of specifying the system operating rules reduces to finding a value of S that minimizes the total expected holding and backorder costs per time unit. Eq. (3.2) reduces to

$$X(t) + Q(t) = S, \quad \text{for } t \geq 0.$$

Clearly $0 \leq Q(t) < S$ indicates inventory on hand while $Q(t) \geq S$ indicates the existence of backorders at time t. Thus, for a conservative system, we need to study only the process $\{Q(t), t \geq 0\}$ to obtain the above expected cost.

The stationary distribution of the process $\{Q(t), t \geq 0\}$ for both a general arrival process and for a Poisson arrival process can be easily obtained following the methods developed in Sections 3.2.3 and 3.4.1, respectively.

3.6.2 Complete Non-recoverability.

This is the classical inventory problem of a consumable item. The inspection-repair loop is eliminated from the system shown in Figure 1.1. All the supplies are received from the external supplier. The system demands can be considered to arise from a single arrival process $\{A_c(t), t \geq 0\}$ with order size distribution $\phi(\cdot)$ as described in Section 3.2. In this case, Eq. (3.2) reduces to

$$Z(t) = X(t) + O(t), \qquad t \geq 0.$$

It can be easily seen that $\Pi(j)$, the stationary distribution of $\{Z(t), t \geq 0\}$, for the (s,S) and the (s,nQ) policies are given by Theorems 3.2 and 3.6, respectively. For obtaining $x(j), j \in Z$, the stationary distribution of $\{X(t), t \geq 0\}$, Eq. (3.5) for an (s,S) policy can be simplified to

$$x_{ij}(t) = \sum_{k \in E} \Pr\{X(t) = j \mid Z(t-\tau) = k;\ Z(0) = i\} \cdot \Pr\{Z(t-\tau) = k \mid Z(0) = i\}$$

$$= \sum_{k \in E} \Pr\{D_c(t-\tau, t) = k-j \mid Z(t-\tau) = k;\ Z(0) = i\}$$

$$\cdot \Pr\{Z(t-\tau) = k \mid Z(0) = i\}.$$

Taking the limit as $t \to \infty$, we get

$$x(j) = \sum_{k \in E} \Pi(k)\, w_{k-j}^{c*}(\tau),$$

where $w_{k-j}^{c*}(\tau)$ is obtained from Eq. (3.11). The resulting expression for $x(j)$ is equivalent to that given by Tijms ([23], pp. 100-101). Similarly, it can be easily shown that for an (s,nQ) policy

$$x(j) = \frac{1}{Q} \sum_{k \in E'} w_{k-j}^{c*}(\tau),$$

where $w_{k-j}^{c*}(\tau)$ is given by Eq. (3.11).

3.7 Certain Long-run Averages.

Combining the results of previous sections, we can determine certain long-run averages which may be used to structure an objective function of total expected cost and to express constraints on system performance measures such as mentioned in Section 1.2.

We first obtain the stationary expected ordering cost per unit time. Let $K_i(t)$ represent the expected number of orders placed in $(0,t]$. Also, referring to Sections 3.2 and 3.3 let

$\beta = a_c$ and $\eta = d_c^m$ for independent demand processes,

and $\beta = a$ and $\eta = d^m$ for dependent demand processes.

From elementary renewal theory [23], we have

$$\lim_{t \to \infty} \frac{K_i(t)}{t} = \frac{\Pi(S)}{\beta}.$$

Furthermore, following the analysis given by Tijms ([23], pp. 113-114), the expected size of the n^{th} $(n \geq 2)$ order is equal to $\frac{\eta}{\Pi(S)}$. If the ordering cost of k units is $K\delta(k) + c \cdot k$ where $\delta(0) = 0$, and $\delta(k) = 1$, for $k > 0$, then the stationary expected ordering cost per unit time is equal to

$$\frac{c\eta}{\beta} + \frac{K\Pi(S)}{\beta} .$$

Under an (s,S) policy the stationary expected number of backorders is given as $\sum_{j=-\infty}^{0} jx(j)$. The stationary expected number of the units held at the supply point is $\sum_{j=1}^{S} jx(j)$. The expected fraction of time the system is out of stock is $1 - \sum_{j=1}^{S} x(j)$.

Similar results can be easily obtained under an (s,nQ) policy.

CHAPTER IV

TWO-ECHELON SYSTEM - UNIT ORDER SIZE

4.1 The Model.

We consider a two-echelon system as described in section 1.1.2 with J bases; the bases are numbered from 1 to J, and the depot is indexed as 0. The failures which generate the system demands occur in a Poisson manner with known parameter λ_j at base j $(j=1,2,\ldots,J)$. Upon such failures, one unit of the item is demanded for replacement. A failed unit turned in at base j is repaired at the base with probability r_j and is shipped to the depot for repair with probability $(1-r_j)\rho$. Thus with probability $(1-r_j)(1-\rho)$, the unit is condemned.

We further specify the following assumptions used in the model.

1. The bases use an $(s-1, s)$ policy for procurement of units from the depot. The depot procurement policy is a general (s,S) policy.
2. Backlogged demands at each location are supplied on a first-come, first-served basis.
3. There are an infinite number of repair facilities at each location. The base repair times R_j and the depot repair time R_0 are deterministic and independent of the arrival process and the number of units in repair. R_0 is the same for all the units received from all the bases.
4. The time to ship a depot repairable unit to the depot from a base is assumed to be negligible. In reality, it can be absorbed in

R_0. The procurement lead times τ_0 for the depot, and τ_j for the bases are deterministic.

Furthermore we assume, as does Simon [20], that $R_0 \leq \tau_0$; that is, the depot repair time does not exceed the depot procurement lead time. The analogous results for the case $R_0 > \tau_0$ can be derived, although this case is less realistic.

We use the following nomenclature.

Nomenclature

λ_j = total demand rate at location j ($j=0,1,\ldots,J$; $j=0$ denotes the depot).

r_j = the probability that a unit that fails at base j will be repaired at base j.

ρ = the probability that a failed unit that is not base repairable will be depot repairable. ρ is the same for all bases.

(s_j-1, s_j) ~ the procurement policy used at base j ($j=1,2,\ldots,J$).

(s_0,S_0) ~ the procurement policy used at the depot.

R_j = the deterministic repair time at location j ($j=0,1,\ldots,J$).

τ_j = the deterministic delivery time from the depot to base j ($j=1,2,\ldots,J$).

τ_0 = the deterministic procurement lead time from the external supplier to the depot.

$D_j(t)$ = the number of units demanded at location j ($j=0,1,\ldots,J$) during the interval $(0,t]$.

$D_j^B(t)$ = the number of units which were declared base repairable at base j during the interval $(0,t]$ ($j=1,2,\ldots,J$).

$D_j^D(t)$ = the number of units sent to the depot for repair from base j during the interval $(0,t]$ ($j=1,2,\ldots,J$).

$D_j^C(t)$ = the number of units condemned at base j during the interval $(0,t]$ $(j=1,2,\ldots,J)$.

$D_j^0(t)$ = the number of units demanded from the depot by base j during the interval $(0,t]$ $(j=1,2,\ldots,J)$.

$D_0^D(t)$ = the number of units received at the depot for repair during the interval $(0,t]$.

$D_0^C(t)$ = the number of units demanded from the depot as a result of condemnations at the bases during the interval $(0,t]$.

$Z_j(t)$ = the inventory position at time t at location j $(j=0,1,\ldots,J)$.

$Q_j(t)$ = the number of units in repair at time t at location j $(j=0,1,\ldots,J)$.

$O_j(t)$ = the number of units on order at time t at location j $(j=0,1,\ldots,J)$.

$B_j(t)$ = the number of backorders at time t at location j $(j=0,1,\ldots,J)$. Negative backorders denote on-hand inventory.

$U_j(t)$ = total number of units on order plus in repair at time t at location j $(j=0,1,\ldots,J)$. Thus, $U_j(t) = O_j(t) + B_j(t)$.

E_0 = $\{s_0+1, s_0+2,\ldots,S_0\}$, the state space of the process $\{Z_0(t), t\geq 0\}$.

$\Pr\{B_j(*)=k\} = \lim_{t\to\infty} \Pr\{B_j(t) = k\}$, $(j=0,1,\ldots,J)$.

$P[n|m] = \dfrac{e^{-m}(m)^n}{n!}$, $n=0,1,\ldots$ (Poisson distribution with mean m).

$N(t_1,t_2) = N(t_2) - N(t_1^+)$ for the process $\{N(t), t\geq 0\}$.

Lower case letters are used to denote a particular realization of a

random variable.

We note the following implications of our assumptions.

(a) Because the bases follow an (s-1, s) policy and the time to place an order from a base to the depot and the time to ship a depot repairable unit to the depot are negligible, we see that for $t \geq 0$

$$D_0^D(t) = \sum_{j=1}^{J} D_j^D(t), \quad \text{and} \quad D_0^C(t) = \sum_{j=1}^{J} D_j^C(t);$$

that is,

$$D_0(t) = D_0^D(t) + D_0^C(t) = \sum_{j=1}^{J} D_j^0(t).$$

(b) For $j=1,2,\ldots,J$; the demand processes $\{D_j^B(t), t \geq 0\}$, $\{D_j^D(t), t \geq 0\}$ and $\{D_j^C(t), t \geq 0\}$ are mutually independent Poisson processes with parameters $\lambda_j^B = r_j\lambda_j$, $\lambda_j^D = (1-r_j)\rho\lambda_j$ and $\lambda_j^C = (1-r_j)(1-\rho)\lambda_j$, respectively. The process $\{D_j^0(t), t \geq 0\}$ is a Poisson process with parameter $\lambda_j^0 = (1-r_j)\lambda_j$.

(c) The depot demand processes $\{D_0^D(t), t \geq 0\}$ and $\{D_0^C(t), t \geq 0\}$ are independent Poisson processes with parameters $\lambda_0^D = \sum_{j=1}^{J} \lambda_j^D$ and $\lambda_0^C = \sum_{j=1}^{J} \lambda_j^C$, respectively. The depot total demand process $\{D_0(t), t \geq 0\}$ is a Poisson process with the parameter $\lambda_0 = \lambda_0^C + \lambda_0^D = \sum_{j=1}^{J} \lambda_j^0$.

(d) Because of the infinite number of repair facilities and constant repair times, the units in repair at base j at time $t(\geq R_j)$ will be due to the base repairable failures occurring only in $(t-R_j, t]$; that is, $Q_j(t) = D_j^B(t-R_j, t)$. Thus for $t \geq R_j$, $Q_j(t)$ is a Poisson variable with mean $\lambda_j^B R_j$, $j=1,2,\ldots,J$. Similarly for $t \geq R_0$, $Q_0(t)$ is a Poisson variable with mean $\lambda_0^D R_0$.

(e) Because depot demands are for a single unit at a time, (s,S) and (s,nQ) policies for the depot are the same with $Q = S-s$.

In view of the problem described in Section 1.2, our goal is to obtain the stationary distributions of the processes $\{Z_j(t), t\geq 0\}$, $\{B_j(t), t\geq 0\}$ and $\{Q_j(t), t\geq 0\}$, for $j=0,1,\ldots,J$. The overall objective of the model is to find the policy values which minimize total expected base backorders. Depot backorders are of interest only insofar as they affect the base backorders. The basic approach for determining the stationary distributions is described in Section 4.2. In Section 4.3, the distributions are obtained. The results for the cases of complete recoverability and compare non-recoverability are derived in Section 4.4.

4.2 The Basic Approach For Stationary Distributions

In this section we describe the basic approach for determining the stationary distributions of the processes $\{Z_j(t), t\geq 0\}$, $\{B_j(t), t\geq 0\}$ and $\{Q_j(t), t\geq 0\}$ $(j=0,1,\ldots,J)$ when the bases use an (s-1, s) procurement policy. The approach will also be applied in the next chapter where the situation of a random order size at the bases is dealt.

4.2.1 The Depot

To obtain the stationary distributions of the procesees $\{Z_0(t), t\geq 0\}$, $\{B_0(t), t\geq 0\}$ and $\{Q_0(t), t\geq 0\}$, we first find the distribution of the processes $\{D_0^D(t), t\geq 0\}$ and $\{D_0^C(t), t\geq 0\}$. The depot can be viewed as a single location system and the results of Section 3.4.1 apply.

4.2.2 The Bases

Because the bases follow an $(s-1, s)$ policy, the inventory position is constant; that is, $Z_j(t) = s_j$ for all $t \geq 0$ and $j=1,2,\ldots,J$.

To find the stationary distribution $B_j(*)$, we first obtain $\Pr\{B_j(t) = b\}$ for $b \in \{-s_j,-s_j+1,\ldots,0,1,\ldots\}$, and then evaluate $\lim_{t\to\infty} \Pr\{B_j(t)=b\}$. The approach is the one given by Kruse and Kaplan [9].

Referring to Figure 4.1, the only units that can arrive at base j from the depot by time t are those on order by time t_3.

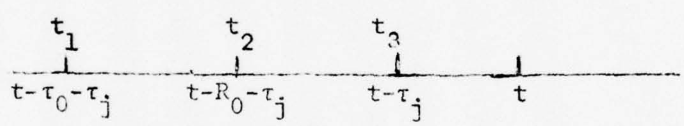


Figure 4.1: The sequence of events at base j.

This depends on the total assets (ready-for-issue units) available at the depot by time t_3, the total demand at the depot during the interval $(t_1,t_3]$ and the sequence of arrivals of requisitions at the depot from the bases during the interval $(t_1,t_3]$. This is so because the units on order through time t_1 will have arrived at the depot from the external supplier by time t_3, and any units not on order by time t_1 will not arrive by time t_3. The total assets available at the depot by time t_3 include the units on hand minus any backorders at time t_1, the units on order at time t_1, the units in repair at time t_1 and the units received for repair during the interval $(t_1,t_2]$. This equals $z_0(t_1) + d_0^D(t_1,t_2)$. Now, the following two mutually exclusive situations are possible:

CASE A: The total depot demand during the interval $(t_1, t_2]$ does not exceed the total assets available by time t_3.

In this case,

$$d_0^D(t_1,t_2) + d_0^C(t_1,t_2) \leq z_0(t_1) + d_0^D(t_1,t_2)$$

or

$$d_0^C(t_1,t_2) \leq z_0(t_1).$$

Thus all the depot demands $d_0(t_1,t_2) = d_0^D(t_1,t_2) + d_0^C(t_1,t_2)$ are satisfied by time t_3. Only the depot demands $d_0(t_2,t_3)$ against the stock of $z_0(t_1) - d_0^C(t_1,t_2)$ units available by time t_3 determine how many demands could possibly remain unsatisfied by time t_3.

CASE B: The total depot demand during the interval $(t_1,t_2]$ exceeds the total available stock by time t_3.

In this case,

$$d_0^D(t_1,t_2) + d_0^C(t_1,t_2) > z_0(t_1) + d_0^D(t_1,t_2)$$

or

$$d_0^C(t_1,t_2) > z_0(t_1).$$

Thus there is no stock available at the depot at time t_2^+ to satisfy the demands $d_0(t_2,t_3)$. Also, there is no guarantee that all the $d_0^D(t_1,t_2)$ demands will be satisfied since this depends on the sequence of arrivals of $d_0^C(t_1,t_2)$ and $d_0^D(t_1,t_2)$. Hence, the total demand $d_0(t_1,t_3)$ drawn against the amount of $z_0(t_1) + d_0^D(t_1,t_2)$ determines how many demands will remain unsatisfied by time t_3.

As a consequence of the $(s_j-1,\ s_j)$ policy at base j, $Z_j(t) = s_j$; that is $-B_j(t) + U_j(t) = s_j$ for all $t \geq 0$. Then for any $b \in \{-s_j, -s_j+1,\ldots,0,1,\ldots\}$ and for any $t \geq \tau_0 + \tau_j$, the event $B_j(t) = b$ occurs if and only if $U_j(t) = s_j + b$. Thus

$$(4.1) \qquad \Pr\{B_j(t) = b\} = \Pr\{U_j(t) = s_j+b\}.$$

In view of the previous discussion, Eq. (4.1) can be rewritten as

$$(4.2) \quad \Pr\{U_j(t) = s_j+b\}$$

$$= \sum_{z_0(t_1)\in E_0} \ \sum_{d_0^C(t_1,t_2)=0}^{\infty} \Big[\Pr\{U_j(t)=s_j+b \,|\, D_0^C(t_1,t_2)=d_0^C(t_1,t_2);$$
$$Z_0(t_1)=z_0(t_1)\}\Pr\{D_0^C(t_1,t_2)=d_0^C(t_1,t_2);\ Z_0(t_1)=z_0(t_1)\}\Big].$$

We investigate $\Pr\{U_j(t) = s_j+b \,|\, D_0^C(t_1,t_2)=d_0^C(t_1,t_2);\ Z_0(t_1)=z_0(t_1)\}$ for the following two mutually exclusive events,

(A) $D_0^C(t_1,t_2) \leq z_0(t_1)$,

and (B) $D_0^C(t_1,t_2) > z_0(t_1)$.

Using the independence of $D_0^C(t_1,t_2)$ and $Z_0(t_1)$ we can express Eq. (4.2) corresponding to events (A) and (B) as follows,

$$(4.3) \quad \Pr\{U_j(t) = s_j+b\}$$

$$= \sum_{z_0(t_1)\in E_0} [\Pr\{U_j(t)=s_j+b\}_A + \Pr\{U_j(t)=s_j+b\}_B]$$

$$\cdot \Pr\{Z_0(t_1)=z_0(t_1)\},$$

where

(4.4) $\Pr\{U_j(t)= s_j+b\}_A$

$$= \sum_{d_0^C(t_1,t_2)=0}^{z_0(t_1)} \Pr\{U_j(t)=s_j+b \mid D_0^C(t_1,t_2)=d_0^C(t_1,t_2);\ Z_0(t_1)=z_0(t_1)\} \cdot \Pr\{D_0^C(t_1,t_2)=d_0^C(t_1,t_2)\},$$

and

(4.5) $\Pr\{U_j(t) = s_j+b\}_B$

$$= \sum_{d_0^C(t_1,t_2)=z_0(t_1)+1}^{\infty} \Pr\{U_j(t)=s_j+b \mid D_0^C(t_1,t_2)=d_0^C(t_1,t_2);\ Z_0(t_1)=z_0(t_1)\} \cdot \Pr\{D_0^C(t_1,t_2)=d_0^C(t_1,t_2)\}.$$

For a given $z_0(t_1)$, Eqs. (4.4) and (4.5) represent $\Pr\{B_j(t) = b\}$ for cases A and B, respectively.

To evaluate Eq. (4.4), let

$$U_j(t) = U_j^1(t) + U_j^2(t),$$

where

$U_j^1(t)$ = the sum of the units in repair at base j at time t and the units for which orders were placed on the depot by base j during the interval $(t_3,t]$,

and

$U_j^2(t)$ = the units ordered from the depot by base j during the interval $(t_2,t_3]$ that remain unfilled by time t_3.

Because the arrival process is Poisson, $U_j^1(t)$ and $U_j^2(t)$ are independent. As mentioned earlier, all the demands levied on the

depot from base j during the interval $(t_3,t]$ will remain on order at time t. Also, the base repairable demands occurring only during the interval $(t-R_j,\ t]$ will be in the repair cycle at base j at time t. Thus, $U_j^1(t) = D_j^B(t-R_j,\ t) + D_j^C(t_3,t) + D_j^D(t_3,t)$ and the probability distribution of $U_j^1(t)$ can be easily obtained. The probability distribution of $U_j^2(t)$ requires considering the sequence of arrivals of requisitions from the bases during the interval $(t_2,t_3]$. Suppose the probability distributions of $U_j^1(t)$ and $U_j^2(t)$ are obtained then Eq. (4.4) can be evaluated by convoluting $U_j^1(t)$ with $U_j^2(t)$. Since $U_j^1(t)$ is independent of $Z_0(t_1)$ and $D_0^C(t_1,t_2)$,

$$(4.6)\quad \Pr\{U_j(t) = s_j+b\}_A = \sum_{d_0^C(t_1,t_2)=0}^{z_0(t_1)} \left[\sum_{d=0}^{s_j+b} \Pr\{U_j^1(t) = s_j+b-d\} \right.$$

$$\left. \cdot \Pr\{U_j^2(t) = d \mid D_0^C(t_1,t_2) = d_0^C(t_1,t_2);\ Z_0(t_1) = z_0(t_1)\} \right]$$

$$\cdot \Pr\{D_0^C(t_1,t_2) = d_0^C(t_1,t_2)\}.$$

To evaluate Eq. (4.5), we note that all the base demands levied on the depot during the interval $(t_2,t]$ remain unfilled by time t. In addition, some demands from base j placed during the interval $(t_1,t_2]$ may remain unfilled by time t_3. Let

$U_j^I(t)$ = the sum of the units in repair at base j at time t and the units for which the orders were placed on the depot by base j during the interval $(t_2,t]$,

and

$U_j^2(t)$ = the units ordered from the depot by base j during the interval $(t_1, t_2]$ that are unfilled by time t_3.

Obviously $U_j(t) = U_j^1(t) + U_j^2(t)$. Because of Poisson arrival process, $U_j^1(t)$ and $U_j^2(t)$ are independent. Here $U_j^1(t) = D_j^B(t-R_j, t) + D_j^C(t_2,t) + D_j^D(t_2,t)$ and we can readily determine the probability distribution of $U_j^1(t)$. The probability distribution of $U_j^2(t)$ involves consideration of the sequence of arrivals of requisitions at the depot during the interval $(t_1,t_2]$. Eq. (4.5) can be obtained through the convolution of $U_j^1(t)$ and $U_j^2(t)$. Again, $U_j^1(t)$ is independent of $Z_0(t_1)$ and $D_0^C(t_1,t_2)$. Thus

$$(4.7)\quad \Pr\{U_j(t) = s_j+b\}_B = \sum_{d_0^C(t_1,t_2)=z_0(t_1)+1}^{\infty} \left[\sum_{d=0}^{s_j+b} \Pr\{U_j^1(t) = s_j+b-d\} \cdot \Pr\{U_j^2(t) = d \mid D_0^C(t_1,t_2) = d_0^C(t_1,t_2);\ Z_0(t_1) = z_0(t_1)\} \right] \cdot \Pr\{D_0^C(t_1,t_2) = d_0^C(t_1,t_2)\}.$$

Upon substituting Eqs. (4.6), (4.7) and the results for $\Pr\{Z_0(t_1) = z_0(t_1)\}$ into Eq. (4.3), we obtain $\Pr\{U_j(t) = s_j+b\}$.

The probability distribution for the process $\{Q_j(t),\ t \geq 0\}$ is easily obtained as

$$\Pr\{Q_j(t) = q\} = \Pr\{D_j^B(t-R_j,\ t) = q\}\ , \quad \text{for } t \geq R_j .$$

4.3 The Stationary Distributions

4.3.1 At The Depot

Since $\{D_0^D(t), t\geq 0\}$ and $\{D_0^C(t), t\geq 0\}$ are independent Poisson processes with parameters λ_0^D and λ_0^C, respectively it follows from Section 3.4.1 that

(4.8) $\lim\limits_{t\to\infty} \Pr\{Z_0(t) = k\} = \Pi_0(k) = \dfrac{1}{S_0 - s_0}$, for $k \in E_0$;

(4.9) $\lim\limits_{t\to\infty} \Pr\{B_0(t) = b_0\} = \dfrac{1}{S_0-s_0} \sum\limits_{k\in E_0} P[k+b_0 \mid \lambda_0^C\tau_0 + \lambda_0^D R_0]$,

for $b_0 = -S_0, -S_0+1, \ldots$;

and

(4.10) $\lim\limits_{t\to\infty} \Pr\{Q_0(t) = q_0\} = P[q_0 \mid \lambda_0^D R_0]$, for $q_0 = 0,1,\ldots$.

4.3.2 At the Bases

To obtain $\Pr\{B_j(t) = b\}$ we compute $\Pr\{U_j(t) = s_j+b\}_A$ and $\Pr\{U_j(t) = s_j+b\}_B$ by evaluating Eqs. (4.6) and (4.7), respectively.

4.3.2.1 Case (A): $\Pr\{U_j(t) = s_j+b\}_A$

Since $U_j^1(t) = D_j^B(t-R_j, t) + D_j^C(t_3,t) + D_j^D(t_3,t)$ and the processes $\{D_j^B(t), t\geq 0\}$, $\{D_j^C(t), t\geq 0\}$ and $\{D_j^D(t), t\geq 0\}$ are independent, $U_j^1(t)$ is a Poisson variable with mean $\lambda_j^B R_j + (\lambda_j^C + \lambda_j^D)\tau_j$; that is,

(4.11) $\Pr\{U_j^1(t) = s_j+b-d\} = P[s_j+b-d \mid \lambda_j^B R_j + (\lambda_j^C + \lambda_j^D)\tau_j]$.

We now proceed to find the probability distribution of $U_j^2(t)$. Suppose we are given $D_0(t_2,t_3) = d_0(t_2,t_3)$ and $D_j^0(t_2,t_3) = d_j^0(t_2,t_3)$. Then $U_j^2(t) = d$ if and only if $d_j^0(t_2,t_3)-d$ units of base j demands are filled by time t_3. In other words, $d_j(t_2,t_3)-d$ of the first $z_0(t_1) - d_0^C(t_1,t_2)$ depot arrivals after time t_2 must come from base j.

Let $RA \equiv [D_j^0(t_2,t_3) = d_j^0(t_2,t_3);\ D_0(t_2,t_3) = d_0(t_2,t_3);\ D_0^C(t_1,t_2) = d_0^C(t_1,t_2);\ Z_0(t_1) = z_0(t_1)]$.

We shall obtain $\Pr\{U_j^2(t)=d \mid D_0^C(t_1,t_2)=d_0^C(t_1,t_2);\ Z_0(t_1) = z_0(t_1)\}$ by enumerating $\Pr\{U_j^2(t) = d \mid RA\}$ over $D_0(t_2,t_3)$ and $D_j^0(t_2,t_3)$.

Because of the properties of the Poisson process [20], we have

$$(4.12)\quad \Pr\{D_j^0(t_2,t_3) = d_j^0(t_2,t_3) \mid D_0(t_2,t_3) = d_0(t_2,t_3)\}$$

$$= \binom{d_0(t_2,t_3)}{d_j^0(t_2,t_3)} [\lambda_j^0/\lambda_0]^{d_j^0(t_2,t_3)}\ [1 - \lambda_j^0/\lambda_0]^{d_0(t_2,t_3)-d_j^0(t_2,t_3)}$$

$$\text{for } d_j^0(t_2,t_3) = 0,1,\ldots,d_0(t_2,t_3).$$

Substituting $\Pr\{D_0(t_2,t_3) = d_0(t_2,t_3)\} = P[d_0(t_2,t_3) \mid \lambda_0 R_0]$, in Eq. (4.12) we get

$$(4.13)\quad \Pr\{D_j^0(t_2,t_3) = d_j^0(t_2,t_3);\ D_0(t_2,t_3) = d_0(t_2,t_3)\}$$

$$= \binom{d_0(t_2,t_3)}{d_j^0(t_2,t_3)} [\lambda_j^0/\lambda_0]^{d_j^0(t_2,t_3)}\ [1 - \lambda_j^0/\lambda_0]^{d_0(t_2,t_3)-d_j^0(t_2,t_3)}$$

$$\cdot\ P[d_0(t_2,t_3) \mid \lambda_0 R_0],$$

$$\text{for } d_j^0(t_2,t_3) \le d_0(t_2,t_3)(\ge 0).$$

To obtain $\Pr\{U_j^2(t) = d | RA\}$ we consider the following two mutually exclusive cases.

(i) $D_0(t_2,t_3) \leq z_0(t_1) - d_0^C(t_1,t_2)$; that is, all the depot demands during interval $(t_2,t_3]$ are satisfied by time t_3. Therefore

$$(4.14)\quad \Pr\{U_j^2(t) = d | RA\} = \begin{cases} 1 & d = 0 ; \\ 0 & d > 0 , \end{cases}$$

(ii) $D_0(t_2,t_3) > z_0(t_1) - d_0^C(t_1,t_2)$; that is, the last $d_0(t_2,t_3) - (z_0(t_1) - d_0^C(t_1,t_2))$ depot demands that arrived during the interval $(t_2,t_3]$ cannot be satisfied by time t_3. Because of a first-come, first-served resupply policy at the depot, using the results of Corollary 1.1 of Appendix A, we have

$$(4.15)\quad \Pr\{U_j^2(t) = d | RA\}$$

$$= \frac{\binom{d_j^0(t_2,t_3)}{d_j^0(t_2,t_3) - d}\binom{d_0(t_2,t_3) - d_j^0(t_2,t_3)}{z_0(t_1) - d_0^C(t_1,t_2) - (d_j^0(t_2,t_3)-d)}}{\binom{d_0(t_2,t_3)}{z_0(t_1) - d_0^C(t_1,t_2)}} .$$

From Eqs. (4.13 - 4.15) we obtain,

$$(4.16)\quad \Pr\{U_j^2(t) = 0 | D_0^C(t_1,t_2) = d_0^C(t_1,t_2);\ Z_0(t_1) = z_0(t_1)\}$$

$$= \sum_{d_0(t_2,t_3)=0}^{z_0(t_1)-d_0^C(t_1,t_2)} P[d_0(t_2,t_3) | \lambda_0 R_0] .$$

$$+\sum_{d_0(t_2,t_3)=z_0(t_1)-d_0^C(t_1,t_2)+1}^{\infty}$$

$$\cdot\left[\sum_{d_j^0(t_2,t_3)=0}^{d_0(t_2,t_3)}\left\{\binom{d_0(t_2,t_3)-d_j^0(t_2,t_3)}{z_0(t_1)-d_0^C(t_1,t_2)-d_j^0(t_2,t_3)}\bigg/\binom{d_0(t_2,t_3)}{z_0(t_1)-d_0^C(t_1,t_2)}\right\}\right.$$

$$\left.\cdot\binom{d_0(t_2,t_3)}{d_j(t_2,t_3)}[\lambda_j^0/\lambda_0]^{d_j^0(t_2,t_3)}[1-\lambda_j^0/\lambda_0]^{d_0(t_2,t_3)-d_j^0(t_2,t_3)}\right]$$

$$\cdot\quad P[d_0(t_2,t_3)|\lambda_0R_0],$$

and

(4.17) $\Pr\{U_j^2(t)=d|D_0^C(t_1,t_2)=d_0^C(t_1,t_2);\ Z_0(t_1)=z_0(t_1)\}$

$$=\sum_{d_0(t_2,t_3)=z_0(t_1)-d_0^C(t_1,t_2)+1}^{\infty}\left[\sum_{d_j^0(t_2,t_3)=0}^{d_0(t_2,t_3)}\left\{\binom{d_j(t_2,t_3)}{d_j(t_2,t_3)-d}\right.\right.$$

$$\left.\cdot\binom{d_0(t_2,t_3)-d_j^0(t_2,t_3)}{z_0(t_1)-d_0^C(t_1,t_2)-d_j^0(t_2,t_3)-d}\bigg/\binom{d_0(t_2,t_3)}{z_0(t_1)-d_0^C(t_1,t_2)}\right\}$$

$$\left.\cdot\binom{d_0(t_2,t_3)}{d_j^0(t_2,t_3)}[\lambda_j^0/\lambda_0]^{d_j^0(t_2,t_3)}[1-\lambda_j^0/\lambda_0]^{d_0(t_2,t_3)-d_j^0(t_2,t_3)}\right]$$

$$\cdot\ P[d_0(t_2,t_3)|\lambda_0R_0],$$

for $d = 1,2,\ldots$.

As mentioned in Section 4.1, $\{D_0^C(t),\ t\geq 0\}$ is a Poisson process with parameter λ_0^C. Therefore

$$(4.18)\quad \Pr\{D_0^C(t_1,t_2) = d_0^C(t_1,t_2)\} = P[d_0^C(t_1,t_2)\,|\,\lambda_0^C(\tau_0-R_0)].$$

Now substituting Eqs. (4.11),(4.16),(4.17) and (4.18) into Eq. (4.6) we obtain $\Pr\{U_j(t) = s_j+b\}_A$. After simplifying (see Appendix B) we get

$$(4.19)\quad \lim_{t\to\infty} \Pr\{U_j(t) = s_j+b\}_A$$

$$= \sum_{d_0^C(t_1,t_2)=0}^{z_0(t_1)} \Bigg\{ P[s_j+b\,|\,\lambda_j^B R_j+(\lambda_j^C+\lambda_j^D)\tau_j]$$

$$\cdot \left[\sum_{d_0(t_2,t_3)=0}^{z_0(t_1)-d_0^C(t_1,t_2)} P[d_0(t_2,t_3)\,|\,\lambda_0 R_0] \right.$$

$$+ \sum_{d_0(t_2,t_3)=z_0(t_1)-d_0^C(t_1,t_2)+1}^{\infty} [1-\lambda_j^0/\lambda_0]^{d_0(t_2,t_3)-z_0(t_1)+d_0^C(t_1,t_2)}$$

$$\left. \cdot P[d_0(t_2,t_3)\,|\,\lambda_0 R_0] \right] + \sum_{d=1}^{s_j+b} P[s_j+b-d\,|\,\lambda_j^B R_j+(\lambda_j^C+\lambda_j^D)\tau_j]$$

$$\cdot \left[\sum_{d_0(t_2,t_3)=z_0(t_1)-d_0^C(t_1,t_2)+d}^{\infty} \binom{d_0(t_2,t_3)-z_0(t_1)+d_0^C(t_1,t_2)}{d} \right.$$

$$\cdot [\lambda_j^0/\lambda_0]^d\, [1-\lambda_j^0/\lambda_0]^{d_0(t_2,t_3)-d-z_0(t_1)+d_0^C(t_1,t_2)}$$

$$\left. \cdot P[d_0(t_2,t_3)\,|\,\lambda_0 R_0] \right] \Bigg\} \cdot P[d_0^C(t_1,t_2)\,|\,\lambda_0^C(\tau_0-R_0)].$$

4.3.2.2 Case (B): $\Pr\{U_j(t) = s_j+b\}_B$

Here $U_j^1(t) = D_j^B(t-R_j, t) + D_j^C(t_2,t) + D_j^D(t_2,t)$; hence, $U_j^1(t)$ is a Poisson variable with mean $\lambda_j^B R_j + (\lambda_j^C + \lambda_j^D)(\tau_j + R_0)$; that is,

$$(4.20)\quad \Pr\{U_j^1(t) = s_j+b-d\} = P[s_j+b-d \mid \lambda_j^B R_j + (\lambda_j^C + \lambda_j^D)(\tau_j + R_0)].$$

To obtain the probability distribution of $U_j^2(t)$ in this case, we follow the approach similar to that used in case (A). Suppose $D_0^D(t_1,t_2) = d_0^D(t_1,t_2)$ and $D_j^0(t_1,t_2) = d_j^0(t_1,t_2)$. Then $U_j^2(t) = d$ if and only if $d_j^0(t_1,t_2)-d$ units of base j demands can be shipped by time t_3; that is, $d_j^0(t_1,t_2)-d$ of the first $z_0(t_1) + d_0^D(t_1,t_2)$ depot arrivals during the interval (t_1,t_2) must come from base j.

Let $RB \equiv \{D_j^0(t_1,t_2) = d_j^0(t_1,t_2);\ D_0^D(t_1,t_2) = d_0^D(t_1,t_2);\ D_0^C(t_1,t_2) = d_0^C(t_1,t_2);\ Z_0(t_1) = z_0(t_1)\}$.

Again because of the properties of the demand process,

$$(4.21)\quad \Pr\{D_j^0(t_1,t_2) = d_j^0(t_1,t_2) \mid D_0^D(t_1,t_2) = d_0^D(t_1,t_2);\ D_0^C(t_1,t_2) = d_0^C(t_1,t_2)\}$$

$$= \binom{d_0^C(t_1,t_2)+d_0^D(t_1,t_2)}{d_j^0(t_1,t_2)} [\lambda_j^0/\lambda_0]^{d_j(t_1,t_2)}$$

$$\cdot\ [1 - \lambda_j^0/\lambda_0]^{d_0^C(t_1,t_2)+d_0^D(t_1,t_2)-d_j^0(t_1,t_2)},$$

$$\text{for } d_j^0(t_1,t_2) = 0,1,\ldots,d_0^C(t_1,t_2)+d_0^D(t_1,t_2).$$

Also,

$$(4.22)\quad \Pr\{D_0^D(t_1,t_2)=d_0^D(t_1,t_2);\ D_0^C(t_1,t_2)=d_0^C(t_1,t_2)\}$$

$$= P[d_0^D(t_1,t_2)|\lambda_0^D(\tau_0-R_0)] \cdot P[d_0^C(t_1,t_2)|\lambda_0^C(\tau_0-R_0)].$$

Using Corollary 1.1 of Appendix A

$$(4.23)\quad \Pr\{U_j^2(t) = d|RB\}$$

$$= \frac{\begin{pmatrix} d_j^0(t_1,t_2) \\ d_j^0(t_1,t_2)-d \end{pmatrix}\begin{pmatrix} d_0^D(t_1,t_2)+d_0^C(t_1,t_2)-d_j^0(t_1,t_2) \\ z_0(t_1)+d_0^D(t_1,t_2)-d_j^0(t_1,t_2)+d \end{pmatrix}}{\begin{pmatrix} d_0^D(t_1,t_2)+d_0^C(t_1,t_2) \\ z_0(t_1)+d_0^D(t_1,t_2) \end{pmatrix}}.$$

Multiplying Eqs. (4.21), (4.22) and (4.23) and then enumerating over $D_j^0(t_1,t_2)$ and $D_0^D(t_1,t_2)$, we obtain $\Pr\{U_j^2(t) = d|D_0^C(t_1,t_2) = d_0^C(t_1,t_2);\ Z_0(t_1) = z_0(t_1)\}$. Substituting this probability and Eq. (4.20) into Eq. (4.7) (see Appendix B) we get

$$(4.24)\quad \lim_{t\to\infty} \Pr\{U_j(t) = s_j+b\}_B$$

$$= \sum_{d=0}^{s_j+b}\left[P[s_j+b-d|\lambda_j^B R_j+(\lambda_j^C+\lambda_j^D)(\tau_j+R_0)]\right.$$

$$\cdot\left[\sum_{d_0^C(t_1,t_2)=z_0(t_1)+d}^{\infty}\right.\left[\sum_{d_0^D(t_1,t_2)=0}^{\infty}\begin{pmatrix} d_0^C(t_1,t_2)-z_0(t_1) \\ d \end{pmatrix}\right.$$

$$\cdot \left[\lambda_j^0/\lambda_0\right]^d \left[1-\lambda_j^0/\lambda_0\right]^{d_0^C(t_1,t_2)-z_0(t_1)-d} p[d_0^D(t_1,t_2)|\lambda_0^D(\tau_0-R_0)]\Big]$$

$$\cdot \left. P[d_0^C(t_1,t_2)|\lambda_0^C(\tau_0-R_0)]\right]\Bigg] .$$

Substituting Eqs. (4.19), (4.24) and (4.8) into Eq. (4.3) we obtain $\Pr\{B_j(*) = b\}$.

For the case where the depot follows an (S_0-1,S_0) policy, $E_0 = \{S_0\}$ and $\Pi(S_0) = 1$. $\Pr\{B_j(*) = b\}$ can now be obtained by substituting $z_0(t_1) = S$ into Eqs. (4.19) and (4.24).

4.4 Special Cases

4.4.1 Complete Recoverability

For the case of complete recoverability, $\rho = 1$; therefore $\lambda_j^C = 0$ $(j=0,1,\ldots J)$ and $\lambda_j^D = (1-r_j)\lambda_j = \lambda_j^0$ $(j=1,2,\ldots,J)$. Consequently $\lambda_0^D = \lambda_0 = \sum_{j=1}^{J} (1-r_j)\lambda_j$. Since there are no condemnations and the system is conservative, no procurement is made by the depot from the external supplier. The inventory position at the depot remains at a constant level S_0 (say), that is, $Z_0(t) = S_0$, for all $t \geq 0$. The stationary distribution of $\{B_0(t), t\geq 0\}$ is obtained in a manner similar to that discussed in Section 3.6.1 and is given by

$$\lim_{t\to\infty} \Pr\{B_0(t) = b_0\} = P[S_0 + b_0|\lambda_0 R_0]$$

$$\text{for } b_0 = -S_0, -S_0 + 1, \ldots, 0, 1, \ldots .$$

Since $D_0^C(t) = 0$ for all $t \geq 0$, case (B) discussed in section 4.2.2 will not arise and consequently Eq. (4.3) reduces to $\Pr\{U_j(t) = s_j+b\} = \Pr\{U_j(t) = s_j+b\}_A$ for $z_0(t_1) = S_0$. Upon substituting $\lambda_0^C = 0$, $\lambda_0^D = \lambda_0$, $d_0^C(t_1,t_2) = 0$ and $z_0(t_1) = S_0$ in Eq. (4.19), we obtain

$$(4.25)\quad \Pr\{B_j(*) = b\} = P[s_j+b \mid \lambda_j^B R_j + \lambda_j^D \tau_j]$$

$$\cdot \left[\sum_{d_0(t_2,t_3)=0}^{S_0} P[d_0(t_2,t_3) \mid \lambda_0 R_0] \right.$$

$$\left. + \sum_{d_0(t_2,t_3)=S_0+1}^{\infty} [1 - \lambda_j^0/\lambda_0]^{d_0(t_2,t_3)-S_0} P[d_0(t_2,t_3) \mid \lambda_0 R_0] \right]$$

$$+ \sum_{d=1}^{s_j+b} P[s_j+b-d \mid \lambda_j^B R_j + \lambda_j^D \tau_j]$$

$$\cdot \left[\sum_{d_0(t_2,t_3)=S_0+d}^{\infty} \binom{d_0(t_2,t_3)-S_0}{d} [\lambda_j/\lambda_0]^d \right.$$

$$\left. \cdot [1 - \lambda_j^0/\lambda_0]^{d_0(t_2,t_3)-d-S_0} P[d_0(t_2,t_3) \mid \lambda_0 R_0] \right] .$$

We note that Eq. (4.24) is equivalent to the result obtained by Simon [20] for the case $\rho = 1$.

4.4.2 Complete Non-recoverability

When an item is consumable, $\rho = 0$ and $r_j = 0$ $(j=1,2,\ldots,J)$. The repair loop is absent at each location in the system and $Q_j(t) = 0$

for $t \geq 0$ and $j = 0,1,\ldots,J$. The stationary distributions of $\{Z_0(t),\ t\geq 0\}$ and $\{B_0(t),\ t\geq 0\}$ are given by Eqs. (4.8) and (4.9), respectively. Setting $r_j = 0$ and $\lambda_0 = \sum_{j=1}^{J} \lambda_j$ in Eq. (4.24), and using Eq. (4.3) we obtain

(4.26) $\Pr\{B_j(*) = b\}$

$$= \sum_{z_0(t_1)\varepsilon E_0} \left[P[s_j+b|\lambda_j\tau_j] \left[\sum_{d_0(t_2,t_3)=0}^{z_0(t_1)} P[d_0(t_2,t_3)|\lambda_0 R_0] \right.\right.$$

$$\left. + \sum_{d_0(t_2,t_3)=z_0(t_1)+1}^{\infty} [1-\lambda_j/\lambda_0]^{d_0(t_2,t_3)-z_0(t_1)} P[d_0(t_2,t_3)|\lambda_0 R_0] \right]$$

$$+ \sum_{d=1}^{s_j+b} P[s_j+b-d|\lambda_j\tau_j]$$

$$\cdot \left[\sum_{d_0(t_2,t_3)=z_0(t_1)+d}^{\infty} \binom{d_0(t_2,t_3)-z_0(t_1)}{d} [\lambda_j/\lambda_0]^d \right.$$

$$\left.\left.\left. \cdot [1-\lambda_j/\lambda_0]^{d_0(t_2,t_3)-d-z_0(t_1)} \cdot P[d_0(t_2,t_3)|\lambda_0 R_0] \right]\right]\right] \frac{1}{(S_0-s_0)} .$$

CHAPTER V

TWO-ECHELON SYSTEM-RANDOM ORDER SIZE

5.1 The Model

In this chapter we study the two-echelon system as described in Section 4.1. Requisitions arrive in a Poisson manner with known parameter λ_j at base j $(j=1,2,\ldots,J)$. Upon arrival of a requisition, a batch containing one or several failed units is turned in, and a like number of new units is demanded for replacement. We shall consider both batch and unit models for the inspection of failed units. In the batch model, a batch as a whole is either base repairable, depot repairable or condemnable, whereas in the unit model, each unit in a batch is inspected independently to find whether it is base repairable, depot repairable or condemnable. We shall use the same assumptions about procurement policies, repair facilities, repair and lead times, as specified in Section 4.1. In addition, partial backlogging of the demands is allowed; that is, upon arrival of a requisition, if the base does not have the number of units demanded, then all the units on hand are supplied while the balance is backlogged. Partial backlogging is also allowed at the depot.

We shall use the following nomenclature in addition to that introduced in Section 4.1

$N_j(t)$ = total number of requisitions that arrived at location j during the interval $(0,t]$ $(j=0,1,\ldots,J)$.

$\phi_j(k)$ = $\Pr\{$upon arrival of a requisition at location j, total number of units demanded $= k\}$, $k\geq 1$, $(j=0,1,\ldots,J)$.

$\phi_j^B(k)$ = Pr{upon arrival of a requisition at base j, the number of base repairable units = k}, $k\geq 1$, $(j=1,2,\ldots,J)$.

$\phi_j^C(k)$ = Pr{upon arrival of a requisition at location j, the number of condemnable units = k}, $k\geq 1$, $(j=0,1,\ldots,J)$.

$\phi_j^D(k)$ = Pr{upon arrival of a requisition at base j, the number of depot repairable units = k}, $k\geq 1$, $(j=1,2,\ldots,J)$.

$CP[k|\lambda t,f] \equiv \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} f^{(n)}(k)$ (compound Poisson distribution with parameter λ and compounding distribution f).

In Section 5.2, we consider the inspection of the failed units under the batch model. In Subsection 5.2.1, we derive the results for the case where order size distribution is the same at all bases; that is, $\phi_1(\cdot) = \phi_2(\cdot) = \ldots = \phi_J(\cdot)$. Subsection 5.2.2 examines the case of different order size distribution at the bases. In Section 5.3, we consider the inspection of the failed units under the unit model.

5.2 The Batch Model

In this model, upon arrival of a requisition at base j, the entire batch of failed units is repaired at the base with probability r_j, is shipped to the depot for repair with probability $(1-r_j)\rho$, or is condemned with probability $(1-r_j)(1-\rho)$. This divides the requisitions into three types: base repairable, depot repairable and condemnable. For $t\geq 0$ and $j=1,2,\ldots,J$, let

$N_j^B(t)$ = the number of requisitions at base j during the interval $(0,t]$ for which the entire batch was declared base repairable,

$N_j^C(t)$ = the number of requisitions at base j during the interval $(0,t]$ for which the entire batch was condemned,

and

$N_j^D(t)$ = the number of requisitions at base j during the interval $(0,t]$ for which the entire batch was sent to the depot for repair.

Obviously, $N_j(t) = N_j^B(t) + N_j^C(t) + N_j^D(t)$, for all $t \geq 0$. Following the arguments used in deriving Eq. (3.52), we see that the processes $\{N_j^B(t), t\geq 0\}$, $\{N_j^C(t), t\geq 0\}$ and $\{N_j^D(t), t\geq 0\}$ are mutually independent Poisson processes with parameters $\lambda_j^B = r_j\lambda_j$, $\lambda_j^C = (1-r_j)(1-\rho)\lambda_j$ and $\lambda_j^D = (1-r_j)\rho\lambda_j$, respectively $(j=1,2,\ldots,J)$. The demand processes $\{D_j^B(t), t\geq 0\}$, $\{D_j^C(t), t\geq 0\}$ and $\{D_j^D(t), t\geq 0\}$ are compound Poisson processes with parameter λ_j^B, λ_j^C and λ_j^D, respectively, and have a common compounding distribution $\phi_j(\cdot)$.

It is clear from the above that the depot receives two distinct types of requisitions from each base. One type requires depot repair for the entire batch, while the other corresponds to a condemned batch of items. For $t\geq 0$, let

$N_j^0(t)$ = total number of requisitions placed at the depot by base j during the interval $(0,t]$ $(j=1,2,\ldots,J)$,

$N_0^C(t)$ = total number of requisitions at the depot during the interval $(0,t]$ as a consequence of condemnations,

and

$N_0^D(t)$ = total number of requisitions at the depot during the interval $(0,t]$ for which the batch of failed units was found depot repairable.

Because the bases use an $(s-1,s)$ policy, $N_j^0(t) = N_j^C(t)+N_j^D(t)$, for all $t \geq 0$ and $j=1,2,\ldots,J$. Consequently, $\{N_j^0, t \geq 0\}$ is a Poisson process with parameter $\lambda_j^0 = \lambda_j^C + \lambda_j^D$. Furthermore, because the bases operate independently, $\{N_0^C(t), t \geq 0\}$ and $\{N_0^D(t), t \geq 0\}$ are Poisson processes with parameters $\lambda_0^C = \sum_{j=1}^{J} \lambda_j^C$ and $\lambda_0^D = \sum_{j=1}^{J} \lambda_j^D$, respectively. Obviously, $N_0(t) = N_0^C(t) + N_0^D(t)$, for all $t \geq 0$. Thus $\{N_0(t), t \geq 0\}$ is a Poisson process with parameters $\lambda_0 = \sum_{j=1}^{J} \lambda_j^0$. Using these results, we now proceed to find the probability distributions for the processes $\{D_0(t), t \geq 0\}$, $\{D_0^C(t), t \geq 0\}$ and $\{D_0^D(t), t \geq 0\}$.

Let $\psi_j(w)$ be the characteristic function of $\phi_j(\cdot)$; that is, $\psi_j(w) = \sum_{k=1}^{\infty} \phi_j(k)e^{iwk}$. Then $\psi_j^0(w)$, the characteristic function of $D_j^0(t)$, is given by [18],

$$\psi_j^0(w) = e^{-\lambda_j^0 t[1-\psi_j(w)]} . \tag{5.1}$$

Thus $\{D_j^0(t), t \geq 0\}$ is a compound Poisson process with parameter λ_j^0 and compounding distribution $\phi_j(\cdot)$. Since the bases operate independently, $D_0(t)$ is the sum of J independent compound Poisson processes. Therefore, $\psi_{D_0(t)}(w)$, the characteristic function of $D_0(t)$, is given by

$$\psi_{D_0(t)}(w) = \prod_{j=1}^{J} \psi_j^C(w)$$

$$= \prod_{j=1}^{J} e^{-\lambda_j^0 t[1-\psi_j(w)]}$$

$$= e^{-\lambda_0 t[1-\frac{1}{\lambda_0}\sum_{j=1}^{J} \lambda_j^0\psi_j(w)]}.$$

Thus $\{D_0(t),\ t\geq 0\}$ is a compound Poisson process with parameter λ_0 and compounding distribution $\phi_0(\cdot)$ whose characteristic function $\psi_0(w)$ is given by

$$\psi_0(w) = \frac{1}{\lambda_0}\sum_{j=1}^{J} \lambda_j^0\psi_j(w).$$

From the additive property of characteristic functions, it follows that

(5.2) $$\phi_0(k) = \frac{1}{\lambda_0}\sum_{j=1}^{J} \lambda_j^0\phi_j(k), \qquad k\geq 1.$$

Similarly, we can show that the demand processes $\{D_0^C(t),\ t\geq 0\}$ and $\{D_0^D(t),\ t\geq 0\}$ are independent compound Poisson processes with parameter λ_0^C and λ_0^D, respectively. Their respective compounding distributions are given by

(5.3) $$\phi_0^C(k) = \frac{1}{\lambda_0^C}\sum_{j=1}^{J} \lambda_j^C\phi_j(k), \qquad k\geq 1,$$

and

(5.4) $$\phi_0^D(k) = \frac{1}{\lambda_0^D} \sum_{j=1}^{J} \lambda_j^D \phi_j(k); \qquad k \geq 1.$$

The depot can now be analyzed as a single location system where recoverable and non-recoverable demand processes are independent compound Poisson processes. Therefore, the results derived in Section 3.4.1 apply. From Eqs. (3.18), (3.46) and (3.20) we have

(5.5) $$\lim_{t\to\infty} \Pr\{Z_0(t)=k \mid Z_0(0)=i,\ i \geq s_0\} \equiv \Pi_0(k) = \begin{cases} \dfrac{m(S-k)}{1+M(S-s-1)} & s+1 \leq k \leq S-1, \\[2ex] \dfrac{1}{1+M(S-s-1)} & k=S; \end{cases}$$

where

$$m(1)=\phi_0^C(1),\ m(k)=\phi_0^C(k)+\sum_{q=1}^{k-1} \phi_0^C(k-q)m(q),\ k \geq 2, \text{ and } M(k) = \sum_{\ell=1}^{k} m(\ell),\ k \geq 1;$$

(5.6) $$\lim_{t\to\infty} \Pr\{Q_0(t) = q_0 \mid Q_0(0)=0\} = CP[q_0 \mid \lambda_0^D R_0,\ \phi_0^D],\quad q_0 \geq 0;$$

and

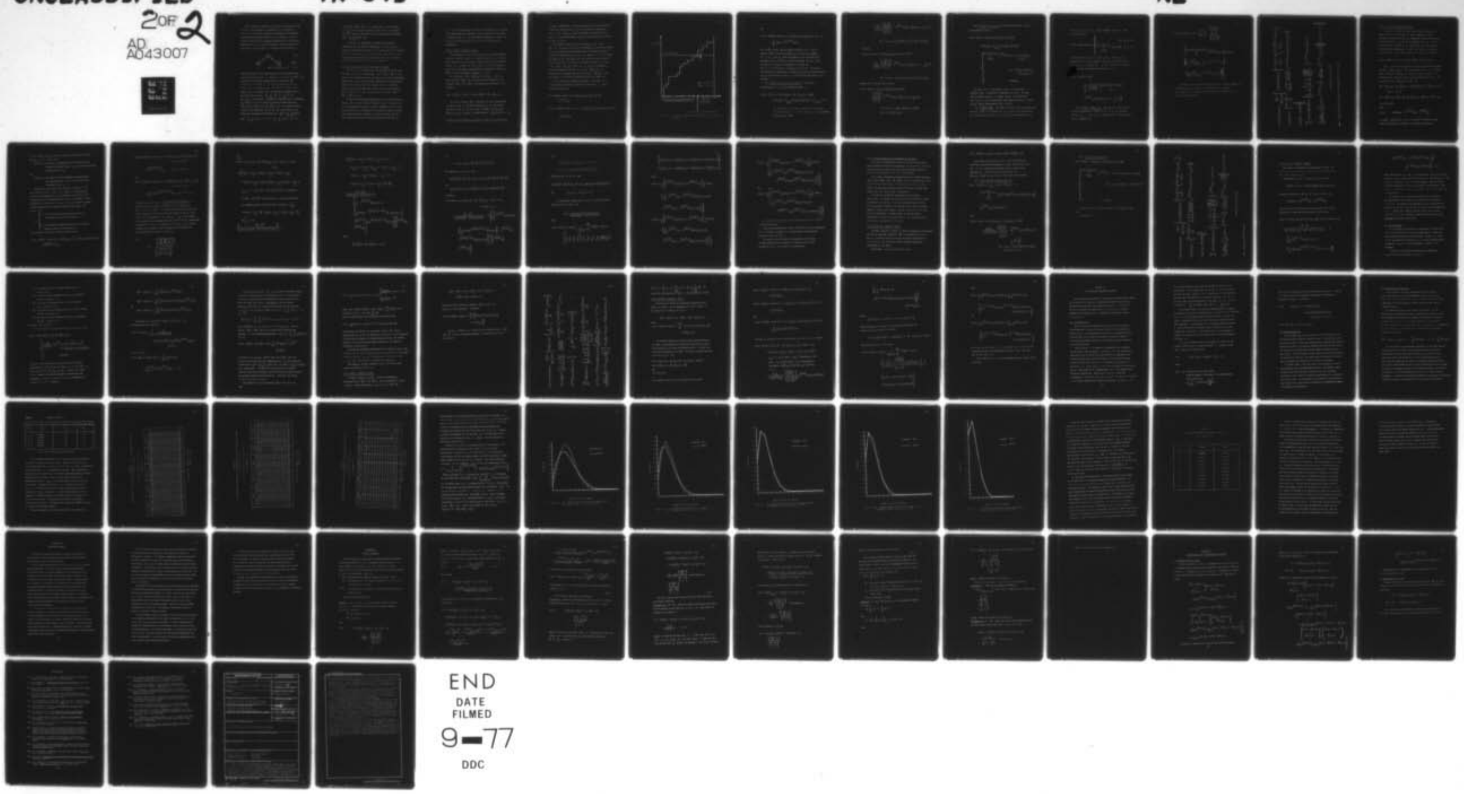
(5.7) $$\lim_{t\to\infty} \Pr\{B_0(t) = b_0 \mid Z_0(0) = i,\ i \geq s_0\}$$

$$= \sum_{k\in E_0} \left\{ \sum_{d_C=0}^{k+b_0} CP[k+b_0-d_C \mid \lambda_0^D R_0,\ \phi_0^D] CP[d_C \mid \lambda_0^C,\ \phi_0^C] \right\} \Pi_0(k)$$

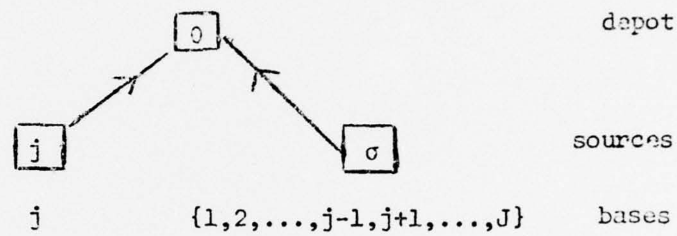
$$b_0 = -S_0, -S_0+1, \ldots, 0, 1, \ldots .$$

AD-A043 007 CORNELL UNIV ITHACA N Y SCHOOL OF OPERATIONS RESEARC--ETC F/G 15/5
AN ANALYSIS OF A TWO-ECHELON INVENTORY SYSTEM FOR RECOVERABLE I--ETC(U)
JUL 77 K SHANKER N00014-75-C-1172
UNCLASSIFIED TR-341 NL

2 OF 2
AD
A043007



Let us consider a subgroup of the bases, containing more than one base. Then following the arguments used in deriving Eqs. (5.2-5.4) we can easily determine the probability distributions for the resultant demand processes of this subgroup. For the purpose of obtaining the stationary distribution of the backorder process at base j, we view the demand at the depot arising from two sources. One, the base j for which the distribution is



being determined, and the other being set of the remaining bases. Let us denote this set by σ, that is, $\sigma = \{1,2,\ldots,j-1,j+1,\ldots,J\}$. Since the bases operate independently, the two sources are independent. For source σ, we shall use notation similar to that used for an individual base. The processes $\{N^B_\sigma(t),\ t\geq 0\}$, $\{N^C_\sigma(t),\ t\geq 0\}$ and $\{N^D_\sigma(t),\ t\geq 0\}$ are mutually independent Poisson processes with parameters $\lambda^B_\sigma = \sum_{i\varepsilon\sigma} \lambda^B_i$, $\lambda^C_\sigma = \sum_{i\varepsilon\sigma} \lambda^C_i$ and $\lambda^D_\sigma = \sum_{i\varepsilon\sigma} \lambda^D_i$, respectively. Following the arguments used in obtaining Eqs. (5.2-5.4) it can be easily seen that the demand processes $\{D^B_\sigma(t),\ t\geq 0\}$, $\{D^C_\sigma(t),\ t\geq 0\}$ and $\{D^D_\sigma(t),\ t\geq 0\}$ are compound Poisson processes with parameters λ^B_σ, λ^C_σ and λ^D_σ, respectively. Their respective compounding distributions are: $\phi^B_\sigma(k) = \frac{1}{\lambda^B_\sigma} \sum_{i\varepsilon\sigma} \lambda^B_i\phi_i(k)$, $\phi^C_\sigma(k) = \frac{1}{\lambda^C_\sigma} \sum_{i\varepsilon\sigma} \lambda^C_i\phi_i(k)$ and $\phi^D(k) = \frac{1}{\lambda^D_\sigma} \sum_{i\varepsilon\sigma} \lambda^D_i\phi_i(k)$; $k\geq 1$. Also,

the process $\{N_\sigma^0(t),\ t\geq 0\}$ is a Poisson process with parameter $\lambda_\sigma^0 = \sum_{i\varepsilon\sigma} \lambda_i^0$ and the demand process $\{D_\sigma^0(t),\ t\geq 0\}$ is a compound Poisson process with parameter λ_σ^0 and compounding distribution $\phi_\sigma^0(k) = \frac{1}{\lambda_\sigma^0} \sum_{i\varepsilon\sigma} \lambda_i^0 \phi_i(k),\ k\geq 1.$

In the next two subsections, we obtain the stationary distribution for the number of backorders at the bases using the approach as described in Section 4.2. In Section 5.2.1, the results are derived for the case where the order size distribution is the same at all bases, while in Section 5.2.2, the case of different order size distributions at the bases is examined.

5.2.1 The Same Order Size Distribution At The Bases

Let $\phi(\cdot)$ be the common order size distribution at the bases. From Eqs. (5.2-5.4), it follows that $\phi(\cdot) = \phi_0^C(\cdot) = \phi_0^D(\cdot) = \phi(\cdot)$; that is, the demand processes at the depot have a common compounding distribution, $\phi(\cdot)$. The stationary distributions of the processes $\{Z_0(t),\ t\geq 0\}$, $\{Q_0(t),\ t\geq 0\}$ and $\{B_0(t),\ t\geq 0\}$ can be obtained from Eqs. (5.5), (5.6) and (5.7), respectively, upon substituting $\phi_0^C(\cdot) = \phi_0^D(\cdot) = \phi(\cdot)$. Also, we note that in this case, $\phi_\sigma^B(\cdot) = \phi_\sigma^C(\cdot) = \phi_\sigma^0(\cdot) = \phi(\cdot)$.

To obtain the stationary distribution of the backorder process $\{B_j(t),\ t\geq 0\}$, we first compute $Pr\{U_j(t) = s_j+b\}$. In order to do so, we consider the two cases A and B as described in Section 4.2. We recall that case A corresponds to the situation where the total depot demand during the interval $(t_1,t_2]$ (see Figure 4.1) does not exceed the total assets available at the depot by time t_3.

Case B, on the other hand, represents the situation where the total depot demand during the interval $(t_1,t_2]$ exceeds the total stock available by time t_3. The probability distribution of $U_j(t)$ for the case A and B are derived in Sections 5.2.1.1 and 5.2.1.2, respectively.

5.2.1.1 Case A: $\Pr\{U_j(t) = s_j+b\}_A$

As mentioned in Section 4.2, in case A all the depot demands during the interval $(t_1,t_2]$ are satisfied by time t_3. Only the depot demands during the interval $(t_2,t_3]$ may remain unsatisfied by time t_3. To obtain $\{\Pr U_j(t) = s_j+b\}$ in case A, we evaluate Eq. (4.6) by computing the probability distributions of $U_j^1(t)$ and $U_j^2(t)$, the two independent components of $U_j(t)$.

By definition, $U_j^1(t) = D_j^B(t-R_j, t) + D_j^O(t_3,t)$. Thus the random variable $U_j^1(t)$ has a compound Poisson distribution with parameter $\lambda_j^B R_j + (\lambda_j^C + \lambda_j^D)\tau_j$ and compounding distribution $\phi(\cdot)$. Therefore,

$$(5.8)\quad \Pr\{U_j^1(t) = s_j+b-d\} = CP[s_j+b-d\,|\,\lambda_j^B R_j + (\lambda_j^C + \lambda_j^D)\tau_j,\ \phi].$$

The random variable $U_j^2(t)$ represents the units ordered from the depot by base j during the interval $(t_2,t_3]$ that remain unfilled by time t_3. We shall obtain $\Pr\{U_j^2(t) = d\,|\,D_0^C(t_1,t_2) = d_0^C(t_1,t_2);\ Z_0(t_1) = z_0(t_1)\}$ by conditioning on $N_i^0(t_2,t_3)$, $i=j,\sigma$. Let

$$RA \equiv \{N_j^0(t_2,t_3) = n_j\ ;\ N_\sigma^0(t_2,t_3) = n_\sigma;\ D_0^C(t_1,t_2) = d_0^C(t_1,t_2);\ Z_0(t_1) = z_0(t_1)\}.$$

To obtain $\Pr\{U_j^2(t)|RA\}$, we need to know the number of requisitions at the depot placed during $(t_2,t_3]$ and completely satisfied by time t_3. Let $N_i'(t_2,t_3)$ denote the number of requisitions from source i that arrived during $(t_2,t_3]$ and are completely satisfied by time t_3, $i=j, \sigma$.

Now, suppose we are given RA and $N_j'(t_2,t_3) = n_j'$. Then $U_j^2(t) = d$ if and only if the sum of the demands due to the unsatisfied $n_j - n_j'$ requisitions and unsatisifed units of possibly a partially satisfied requisition (if from source j) equals d. Let EX denote the number of the units supplied to the requisition whose demand is only partially met. The range of the random variable EX is from 0 to $z_0(t_1) - d_0^C(t_1,t_2)$. When $EX = 0$, there is no partially satisfied requisition and when $EX = z_0(t_1) - d_0^C(t_1,t_2)$, no requisition is completely satisfied and $z_0(t_1) - d_0^C(t_1,t_2)$ units are supplied to the first requisition (if any) (see Figure 5.1)

Let us introduce an indicator variable I such that $I = i$ if the partially satisfied requisition is from source i, $i=j, \sigma$. We have the following.

(5.9) $$\Pr\{U_j^2(t) = d|RA;\ EX = 0;\ N_j'(t_2,t_3) = n_j';\ N_\sigma'(t_2,t_3) = n_\sigma'\}$$
$$= \phi^{(n_j - n_j')}(d);$$

(5.10) $$\Pr\{U_j^2(t) = d|RA;\ EX = e > 0;\ N_j'(t_2,t_3) = n_j';\ N_\sigma'(t_2,t_3) = n_\sigma';\ I = \sigma\}$$
$$= \phi^{(n_j - n_j')}(d);$$

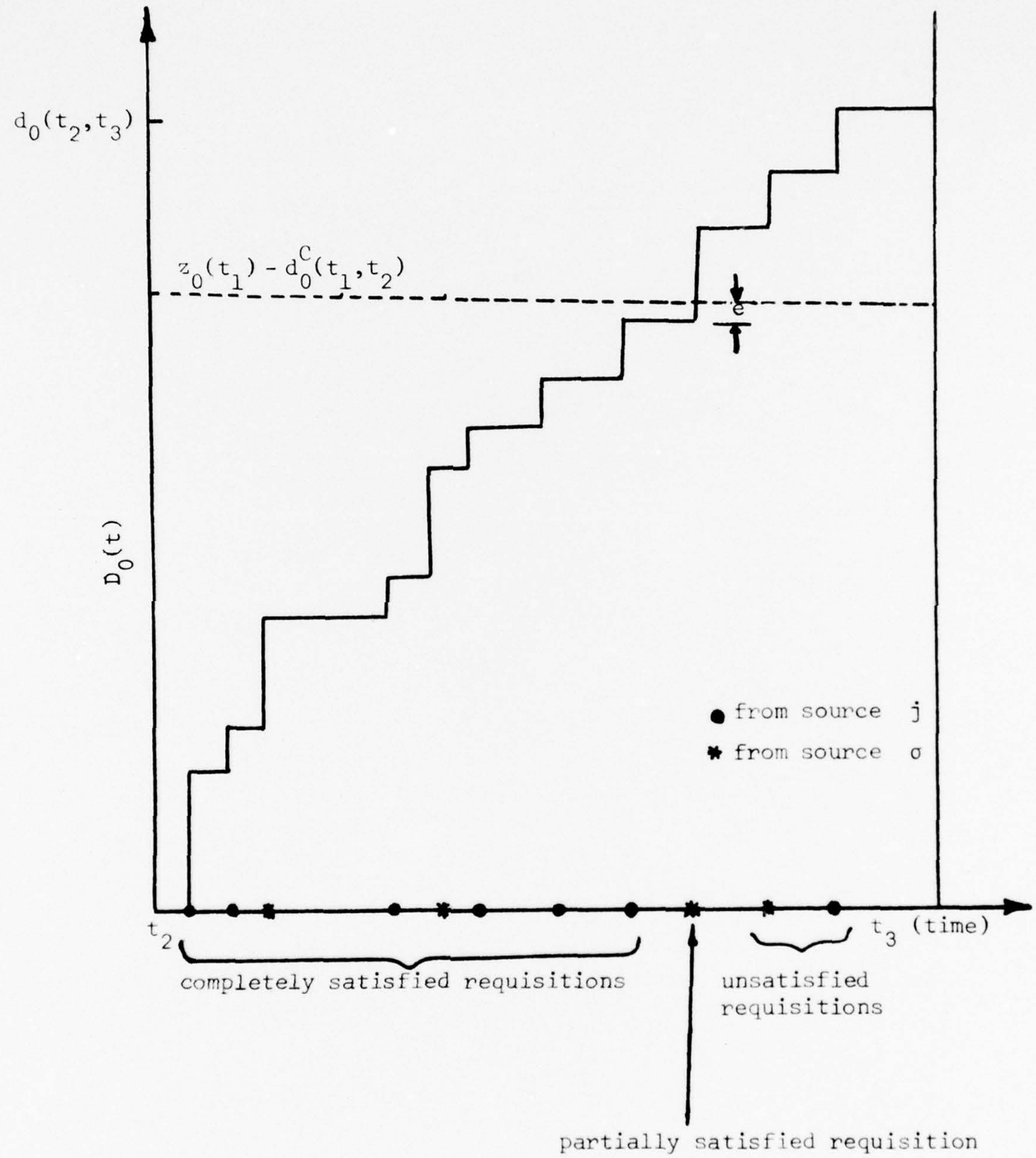


Figure 5.1: A sample realization of EX, $N_j'(t_2,t_3)$ and $N_\sigma'(t_2,t_3)$ given RA.

and

$$(5.11)\quad \Pr\{U_j^2(t) = d \mid RA;\ EX = e > 0;\ N_j'(t_2,t_3) = n_j';\ N_\sigma'(t_2,t_3) = n_\sigma';\ I = j\}$$

$$= \sum_{k>0}^{d} \phi(k+e) \cdot \phi^{(n_j - n_j' - 1)}(d-k).$$

The $\Pr\{U_j^2(t) = d \mid RA\}$ can be obtained from Eqs. (5.9 - 5.11) by knowing $\Pr\{EX = e(\geq 0);\ I=i;\ N_j'(t_2,t_3) = n_j';\ N_\sigma'(t_2,t_3) = n_\sigma' \mid RA\}$, for $i = j,\sigma$. Let Y_k denote the number of units demanded from the depot by the k^{th} $(k \geq 1)$ requisition during $(t_2,t_3]$. Using the results of Theorem 1 of Appendix A, we obtain the $\Pr\{EX = e,\ I=i,\ N_j'(t_2,t_3) = n_j';\ N_\sigma'(t_2,t_3) = n_\sigma' \mid RA\}$. We consider the following two situations. One where not all the requisitions during $(t_2,t_3]$ are satisfied by time t_3; and the other, where they are all satisfied.

(i) $\underline{0 \leq n_j' + n_\sigma' < n_j + n_\sigma;\ \ n_j + n_\sigma \geq 1}$, not all the requisitions in $(t_2,t_3]$ are satisfied by time t_3.

$$(5.12)\quad \Pr\{EX = e;\ I=j;\ N_j'(t_2,t_3) = n_j';\ \ N_\sigma'(t_2,t_3) = n_\sigma' \mid RA\}$$

$$= \Pr\{(Y_1 + Y_2 + \ldots + Y_{n_j' + n_\sigma'}) = z_0(t_1) - d_0^C(t_1,t_2) - e;\ Y_{n_j' + n_\sigma' + 1} > e;$$

out of the first $(n_j' + n_\sigma')$ requisitions at the depot. n_j' are from source j, and the $(n_j' + n_\sigma' + 1)$th requisition is from source $j \mid RA\}$

$$= \frac{n_j'+1}{n_j'+n_\sigma'+1} \frac{\binom{n_j}{n_j'+1}\binom{n_\sigma}{n_\sigma'}}{\binom{n_j+n_\sigma}{n_j'+n_\sigma'+1}} \cdot \phi^{(n_j'+n_\sigma')}(z_0(t_1) - d_0^C(t_1,t_2) - e) \cdot \sum_{\ell>e} \phi(\ell),$$

for $e = 0,1,\ldots,z_0(t_1) - d_0^C(t_1,t_2) - (n_j' + n_j')$; $0 \le n_j' \le n_j$.

Similarly,

(5.13) $\Pr\{EX = e;\ I=\sigma;\ N_j'(t_2,t_3) = n_j';\ N_\sigma'(t_2,t_3) = n_\sigma' | RA\}$

$$= \frac{n_\sigma'+1}{n_j'+n_\sigma'+1} \frac{\binom{n_j}{n_j'}\binom{n_\sigma}{n_\sigma'+1}}{(n_j + n_\sigma)} \cdot \phi^{(n_j'+n_\sigma')}(z_0(t_1) - d_0^C(t_1,t_2) - e) \cdot \sum_{\ell>e} \phi(\ell),$$

for $e = 0,1,\ldots,z_0(t_1) - d_0^C(t_1,t_2) - (n_j' + n_\sigma')$; $0 \le n_\sigma' \le n_\sigma$.

Summing Eqs. (5.12) and (5.13) we have

(5.14) $\Pr\{EX = e;\ N_j'(t_2,t_3) = n_j';\ N_\sigma'(t_2,t_3) = n_\sigma' | RA\}$

$$= \frac{\binom{n_j}{n_j'}\binom{n_\sigma}{n_\sigma'}}{\binom{n_j+n_\sigma}{n_j'+n_\sigma'}} \phi^{(n_j'+n_\sigma')}(z_0(t_1) - d_0^C(t_1,t_2) - e) \cdot \sum_{\ell>e} \phi(\ell),$$

for $e = 0,1,\ldots,z_0(t_1) - d_0^C(t_1,t_2) - (n_j'+n_\sigma')$,

and $0 \le n_j' + n_\sigma' < n_j + n_\sigma$.

(ii) $\underline{n_j' + n_\sigma' = n_j + n_\sigma\ (\geq 0)}$ all the requisitions in $(t_2,t_3]$ are satisfied by time t_3.

$$(5.15)\quad \Pr\{EX = e;\ N_j'(t_2,t_3) = n_j';\ N_\sigma'(t_2,t_3) = n_\sigma' \mid RA\}$$

$$= \Pr\{(Y_1+Y_2+\ldots+Y_{n_j'+n_\sigma'}) \leq z_0(t_1) - d_0^C(t_1,t_2)\}$$

$$= \begin{cases} \displaystyle\sum_{\ell=n_j'+n_\sigma'}^{z_0(t_1)-d_0^C(t_1,t_2)} \phi^{(n_j'+n_\sigma')}(\ell), & \text{for } e = 0, n_j'=n_j, n_\sigma'=n_\sigma,\ n_j+n_\sigma \geq 1; \\ 1, & \text{for } e = z_0(t_1) - d_0^C(t_1,t_2),\ n_j' = n_\sigma' = n_j = n_\sigma = 0; \\ 0, & \text{otherwise} \end{cases}$$

From Eqs. (5.9 - 5.11) and Eqs. (5.12 - 5.15) we obtain $\Pr\{U_j^2(t) = d \mid RA\}$. We consider the following two cases: one where all the demands from base j during (t_2,t_3) are satisfied by time t_3 $(d=0)$; and the other, where some demands from base j during $(t_2,t_3]$ remain unsatisfied by time t_3 $(d \geq 1)$.

(i) $\underline{\Pr\{U_j^2(t) = 0 \mid RA\}}$. From Eqs. (5.9) and (5.10) we conclude that given RA, $U_j^2(t) = 0$ if and only if $n_j' = n_j\ (\geq 0)$. For the case when all the requisitions at depot in $(t_2,t_3]$ are satisfied by time t_3,

that is, $n_j' + n_\sigma' = n_j + n_\sigma$, then $\Pr\{U_j^2(t) = 0|RA\} = 1$. From Eq. (5.15) we have

$$(5.16)\quad \Pr\{U_j^2(t)=0|RA\} = \begin{cases} 1 & , \text{for } n_j+n_\sigma = 0 . \\ \sum_{\ell=n_j+n_\sigma}^{z_0(t_1)-d_0^C(t_1,t_2)} \phi^{(n_j+n_\sigma)}; & \text{for } n_j+n_\sigma \geq 1 . \end{cases}$$

On the other hand, when $n_j' + n_\sigma' < n_j + n_\sigma$, that is, not all requisitions at the depot during $(t_2,t_3]$ are satisfied by time t_3, then $n_j = n_j'$ implies that $n_\sigma' < n_\sigma$. From Eq. (5.13), after simplifying the combinatorial expressions we have

$$(5.17)\quad \Pr\{U_j^2(t) = 0|RA\}$$

$$= \sum_{n_\sigma'=0}^{n_\sigma-1} \frac{n_\sigma!(n_j+n_\sigma')!}{n_\sigma'!(n_j+n_\sigma)!} \left\{ \sum_{e=0}^{z_0(t_1)-d_0^C(t_1,t_2)-(n_j+n_\sigma')} \phi^{(n_j+n_\sigma')}(z_0(t_1)-d_0^C(t_1,t_2) - e) \cdot \sum_{\ell>e} \phi(\ell) \right\} .$$

(ii) $\underline{\Pr\{U_j^2(t) = d|RA\},\ d \geq 1}$ From Eqs. (5.9) and (5.10) it is clear that given RA, $U_j^2(t) = d > 0$ if $n_j' < n_j$ and thus $n_j' + n_\sigma' < n_j + n_\sigma$ for $n_j \geq 1$. From Eqs. (5.9 - 5.11) and (5.12 - 5.14), it follows that

$$(5.18)\quad \Pr\{U_j^2(t) = d \mid RA\} = \sum_{n_j'=0}^{n_j-1} \sum_{n_\sigma'=0}^{n_\sigma} \frac{\binom{n_j}{n_j'}\binom{n_\sigma}{n_\sigma'}}{\binom{n_j+n_\sigma}{n_j'+n_\sigma'}}$$

$$\cdot \left\{ \sum_{e=0}^{z_0(t_1)-d_0^C(t_1,t_2)-(n_j'+n_\sigma')} \phi^{(n_j'+n_\sigma')}(z_0(t_1)-d_0^C(t_1,t_2)-e) \right.$$

$$\cdot \left\{ \left(\frac{n_j-n_j'}{(n_j+n_\sigma)-(n_j'+n_\sigma')} \sum_{k>0}^{d} \phi(k+e)\phi^{(n_j-n_j'-1)}(d-k) \right) \right.$$

$$\left. \left. + \left(\frac{n_\sigma-n_\sigma'}{(n_j+n_\sigma)-(n_j'+n_\sigma')} \phi^{(n_j-n_j')}(d) \sum_{\ell>e} \phi(\ell) \right) \right\} \right\}.$$

From Eqs. (5.16), (5.17) and (5.18) we obtain $\Pr\{U_j^2(t) = d \mid D_0^C(t_1,t_2) = d_0^C(t_1,t_2);\ Z_0(t_1) = z_0(t_1)\}$ by enumerating over $N_j(t_2,t_3)$ and $N_\sigma(t_2,t_3)$. Substituting the resulting expression and Eq. (5.8) into Eq. (4.6) we obtain

$$
(5.19)\quad \Pr\{U_j(t)=s_j+b\}_A = \sum_{d_0^C(t_1,t_2)=0}^{z_0(t_1)} \Bigg[CP[s_j+b|\lambda_j^B R_j+\lambda_j^0\tau_j,\phi] \cdot \Bigg\{ \sum_{n_j+n_\sigma=0}^{z_0(t_1)-d_0^C(t_1,t_2)} \Bigg(\sum_{\ell=n_j+n_\sigma}^{z_0(t_1)-d_0^C(t_1,t_2)} \phi^{(n_j+n_\sigma)}(\ell) \Bigg)
$$

$$
\cdot P[n_j+n_\sigma|\lambda_0 R_0] \quad + \Bigg\{ \sum_{n_j=0}^{z_0(t_1)-d_0^C(t_1,t_2)} \sum_{n_\sigma=1}^{\infty} \Bigg\{ \sum_{n_\sigma'=0}^{n_\sigma-1} \frac{n_\sigma!(n_j+n_\sigma')!}{n_\sigma'!(n_j+n_\sigma)!} \sum_{e'} \phi^{(n_j+n_\sigma')}(z_0(t_1)-d_0^C(t_1,t_2)-e) \cdot \sum_{\ell>e} \phi(\ell) \Bigg\}
$$

$$
\cdot P[n_\sigma|\lambda_\sigma^0 R_0] \cdot P[n_j|\lambda_j^0 R_0] \Bigg\}\Bigg\} \qquad +
$$

$$
\sum_{d=1}^{s_j+b} CP[s_j+b-d|\lambda_j^B R_j+\lambda_j^0\tau_j,\phi] \cdot \Bigg\{ \sum_{n_j=1}^{\infty} \sum_{n_\sigma=0}^{\infty} \Bigg\{ \sum_{n_j'=0}^{n_j-1} \sum_{n_\sigma'=0}^{n_\sigma} \frac{\binom{n_j}{n_j'}\binom{n_\sigma}{n_\sigma'}}{\binom{n_j+n_\sigma}{n_j'+n_\sigma'}} \cdot \Bigg\{ \sum_{e} \phi^{(n_j'+n_\sigma')}(z_0(t_1)-d_0^C(t_1,t_2)-e)
$$

$$
\Bigg\{ \frac{n_j-n_j'}{n_j+n_\sigma-n_j'-n_\sigma'} \Bigg(\sum_{k>0}^{d} \phi(k+e) \cdot \phi^{(n_j-n_j'-1)}(d-k) \Bigg) + \frac{n_\sigma-n_\sigma'}{n_j+n_\sigma-n_j'-n_\sigma'} \Bigg(\phi^{(n_j-n_j')}(d) \sum_{\ell>e} \phi(\ell) \Bigg) \Bigg\}\Bigg\}\Bigg\} P[n_\sigma|\lambda_\sigma^0 R_0] \cdot P[n_j|\lambda_j^0 R_0] \Bigg\}\Bigg]
$$

$$
\cdot\, CP[d_0^C(t_1,t_2)|\lambda_0^C(\tau_0-R_0),\phi_0^C].
$$

where $0 \le e' \le z_0(t_1)-d_0^C(t_1,t_2)-(n_j+n_\sigma')$ and $0 \le e \le z_0(t_1)-d_0^C(t_1,t_2)-(n_j'+n_\sigma)$.

5.2.1.2 Case (B): $\Pr\{U_j(t) = s_j+b\}_B$

In this case, none of the depot demands during $(t_2,t_3]$ are filled by time t_3. The stock of $z_0(t_1) + d_0^D(t_1,t_2)$ units available at the depot by time t_3 determines how many requisitions at the depot during $(t_1,t_2]$ can be filled by time t_3. To obtain $\Pr\{U_j(t) = s_j+b\}$ in case B, we evaluate Eq. (4.7) by computing the probability distributions of $U_j^1(t)$ and $U_j^2(t)$ as described in Section 4.2. Here, $U_j^1(t) = D_j^B(t-R_j, t) + D_j^C(t_2,t)$; therefore,

$$(5.20)\quad \Pr\{U_j^1(t) = s_j+b-d\} = CP[s_j+b-d \mid \lambda_j^B R_j + \lambda_j^0(\tau_j + R_0), \phi].$$

$U_j^2(t)$, in this case, represents the units ordered from the depot by base j during (t_1,t_2) that are unfilled by time t_3. We shall obtain $\Pr\{U_j^2(t) = d \mid D_0^C(t_1,t_2) = d_0^C(t_1,t_2);\ Z_0(t_1) = z_0(t_1)\}$ by conditioning on $N_i^C(t_1,t_2)$, $N_i^D(t_1,t_2)$ and $D_i^D(t_1,t_2)$, $i = j,\sigma$. We temporarily denote $d_0^C(t_1,t_2)$ by d_0^C. Let

$$RB1 \equiv \{N_i^C(t_1,t_2) = n_j^C;\ N_j^D(t_1,t_2) = n_j^D;\ N_\sigma^C(t_1,t_2) = n^C;\ N_\sigma^D(t_1,t_2) = n^D;$$
$$Z_0(t_1) = z_0(t_1)\},$$

$$RB2 \equiv \{D_j^C(t_1,t_2) + D_\sigma^C(t_1,t_2) = d_0^C;\ D_j^D(t_1,t_2) + D_\sigma^D(t_1,t_2) = d_0^D\},\ \text{and}$$

$$RB = RB1 \cup RB2.$$

Then

$$(5.21)\qquad \Pr\{RB2 \mid RB1\} = \phi^{(n_j^C+n_\sigma^C)}(d_0^C)\cdot\phi^{(n_j^D+n_\sigma^D)}(d_0^D).$$

To obtain $\Pr\{U_j^2(t) \mid RB\}$, we need to consider the number and type (depot repairable or condemnable) of requisitions placed at

the depot during $(t_1,t_2]$ from each source and completely satisfied by time t_3. For $i = j,\sigma$, let

$N_i^{'C}(t_1,t_2)$ = the number of condemnation type requisitions that arrived at the depot during $(t_1,t_2]$ and are completely satisfied by time t_3;

and

$N_i^{'D}(t_1,t_2)$ = the number of depot repairable type requisitions that arrived during $(t_1,t_2]$ and are completely satisfied by time t_3.

Now suppose we are given RB and know that $N_i^{'C}(t_1,t_2) = n_i^{'C}$ and $N_i^{'D}(t_1,t_2) = n_i^{'D}$, $i = j,\sigma$. Then $U_j^2(t) = d$ if and only if the sum of the demands due to the unsatisfied $(n_j^C + n_j^D - n_j^{'C} - n_j^{'D})$ requisitions and unsatisfied units of possibly a partially satisfied requisition (if from source j) equals d. Let EX denote the number of units supplied to the partially satisfied requisition. For $i = j,\sigma$, let

$$I = \begin{cases} iC, & \text{if the partially satisfied requisition is of condemnation type and is from source } i, \\ iD, & \text{if the partially satisfied requisition is of depot repairable type and is from source } i. \end{cases}$$

Then similar to Eqs. (5.9), (5.10) and (5.11) we have

$$(5.22)\quad \Pr\{U_j^2(t) = d \mid RB;\ EX = 0;\ (N_i^{'C}(t_1,t_2) = n_i^{'C};\ N_i^{'D}(t_1,t_2) = n_i^{'D},\ i=j,\sigma)\}$$
$$= \phi^{(n_j^C+n_j^D-n_j^{'C}-n_j^{'D})}(d);$$

$$(5.23)\quad \Pr\{U_j^2(t) = d \mid RB;\ EX = e > 0;\ (N_i^{'C}(t_1,t_2) = n_i^{'C},\ N_i^{'D}(t_1,t_2) = n_i^{'D};\ i = j,\ \sigma);\ I = \delta\}$$

$$= \phi^{(n_j^C + n_j^D - n_j^{'C} - n_j^{'D})}(d), \qquad \text{for} \quad \delta = \sigma C,\ \sigma D;$$

and

$$(5.24)\quad \Pr\{U_j^2(t) = d \mid RB;\ EX = e > 0;\ (N_i^{'C}(t_1,t_2) = n_i^{'C};\ N_i^{'D}(t_1,t_2) = n_i^{'D},\ i = j,\ \sigma);\ I = \delta\}$$

$$= \sum_{k>0}^{d} \phi(k+e)\phi^{(n_j^C + n_j^D - n_j^{'C} - n_j^{'D} - 1)}(d-k), \quad \text{for} \quad \delta = jC,\ jD.$$

In this case, we note that $0 \le n_j^{'C} + n_j^{'D} + n_\sigma^{'C} + n_\sigma^{'D} < n_j^C + n_j^D + n_\sigma^C + n_\sigma^D$. We now need to obtain $\Pr\{EX = e,\ (N_i^{'C}(t_1,t_2) = n_i^{'C};\ N_i^{'D}(t_1,t_2) = n_i^{'D},\ i = j,\ \sigma);\ I = \delta \mid RB\}$. This will be done following the approach used to obtain Eq. (5.12), and using the results of Theorem 2 and Corollary 2.2 of Appendix A. Let $\{Y_1, Y_2, \ldots, Y_{n_i^C}\}$ be the sequence of the number of units demanded from the depot by the n_i^C requisitions from source i $(i = j,\ \sigma)$. Similarly, let $\{Y_1, Y_2, \ldots, Y_{n_i^D}\}$ be the sequence of the number of units demanded from the depot by the n_i^D requisitions from source i $(i = j,\ \sigma)$.

Let

$$PS = \frac{\binom{n_j^C}{n_j^{'C}}\binom{n_j^D}{n_j^{'D}}\binom{n_\sigma^C}{n_\sigma^{'C}}\binom{n_\sigma^D}{n_\sigma^{'D}}}{\binom{n_j^C + n_j^D + n_\sigma^C + n_\sigma^D}{n_j^{'C} + n_j^{'D} + n_\sigma^{'C} + n_\sigma^{'D}}}$$

Then

(5.25)

$$\Pr\{EX = e;\ N_i^{'C}(t_1,t_2) = n^{'C};\ N_i^{'D}(t_1,t_2) = n_i^{'D},\ \ i=j,\sigma);\ I = kC|RB\},$$

$$k = j,\sigma$$

$$= \Pr\Big\{ (Y_1+Y_2+\ldots+Y_{n_j^{'C}}) + (Y_1+Y_2+\ldots+Y_{n_j^{'D}}) + (Y_1+Y_2+\ldots+Y_{n_\sigma^{'C}})$$

$$+ (Y_1+Y_2+\ldots+Y_{n_\sigma^{'D}}) = z_0(t_1) + (Y_1+Y_2+\ldots+Y_{n_j^D}) + (Y_1+Y_2+\ldots+Y_{n_\sigma^D}) - e;$$

$Y_{n_k^{'C}+1} > e$; out of first $(n_j^{'C} + n_j^{'D} + n_\sigma^{'C} + n_\sigma^{'D})$ requisitions

at depot, $(n_j^{'C} + n_j^{'D})$ are from source j; the next requisition

is condemnation type and is from source $k \,|\, (Y_1+Y_2+\ldots+Y_{n_j^D})$

$$+ (Y_1+Y_2+\ldots+Y_{n_j^D}) = d_0^D;\ \ (Y_1+Y_2+\ldots+Y_{n_j^C}) + (Y_1+Y_2+\ldots+Y_{n_\sigma^C}) = d_0^C;$$

$$RB1 \Big\};\ \ k = j,\sigma$$

$$= \left[\frac{(n_k^C - n_k^{'C})}{(n_j^C + n_j^D + n_\sigma^C + n_\sigma^D) - (n_j^{'C} + n_j^{'D} + n_\sigma^{'C} + n_\sigma^{'D})} \right] \cdot PS$$

$$\cdot \Pr\Big\{ (Y_1+Y_2+\ldots+Y_{n_j'^C}) + (Y_1+Y_2+\ldots+Y_{n_\sigma'^C}) = z_0(t_1)$$

$$+ (Y_{n_j'^D+1} + \ldots + Y_{n_j^D}) + (Y_{n_\sigma'^D+1} + \ldots + Y_{n_\sigma^D}) - e;\; Y_{n_k'^C+1} > e \,\Big|$$

$$(Y_1+Y_2+\ldots+Y_{n_j^D}) + (Y_1+Y_2+\ldots+Y_{n_\sigma^D}) = d_0^D;$$

$$(Y_1+Y_2+\ldots+Y_{n_j^C}) + (Y_1+Y_2+\ldots+Y_{n_\sigma^C}) = d_0^C;\ \mathrm{RB1}\Big\}$$

$$= \frac{(n_k^C - n_k'^C)}{(n_j^C+n_j^D+n_\sigma^C+n_\sigma^D) - (n_j'^C+n_j'^D+n_\sigma'^C+n_\sigma'^D)}$$

$$\cdot \left\{ \sum_{k_1} \phi^{(n_j'^C + n_\sigma'^C)}(z_0(t_1) + k_1 - e) \right.$$

$$\cdot \left\{ \sum_{k_2} \phi(e+k_2) \cdot \phi^{(n_j^C + n_\sigma^C - n_\sigma'^C - n_\sigma^C - 1)}(d_0^C - z_0(t_1) - k_1 - k_2) \right\}$$

$$\left. \cdot \left(\phi^{(n_j'^D + n_\sigma'^D)}(k_1)\, \phi^{(n_j^D + n_\sigma^D - n_j'^D - n_\sigma'^D)}(d_0^C - k_1) \right) \right\} \Bigg/ \left\{ \phi^{(n_j^D+n_\sigma^D)}(d_0^D) \cdot \phi^{(n_j^C+n_\sigma^C)}(d_0^C) \right\},$$

where

$$n_k'^D \le n_k^D\ (\ge 0),\quad n_k'^C < n_k^C\ (\ge 1);\quad k = j,\sigma$$

and

$$e = 0,1,\ldots,z_0(t_1) + d_0^D - (n_j^{'C} + n_j^{'D} + n_\sigma^{'C} + n_\sigma^{'D}).$$

The ranges for k_1 and k_2 are,

$$\max\{n_j^{'D} + n_\sigma^{'D},\ n_j^{'C} + n_\sigma^{'C} - z_0(t_1) + e\} \le k_1 \le d_0^D - (n_j^D + n_\sigma^D - n_j^{'D} - n_\sigma^{'D}),$$

and

$$1 \le k_2 \le \min\{d_0^C - z_0(t_1) - k_1 - (n_j^C + n_\sigma^C - n_j^{'C} - n_\sigma^{'C}),\ d_0^C - e - (n_j^C + n_\sigma^C - n_j^{'C} - n_\sigma^{'C})\}.$$

Similarly,

$$(5.26)\ \Pr\{Ex = e;\ (N_i^{'C}(t_1,t_2) = n_i^{'C};\ N_i^{'D}(t_1,t_2) = n_i^{'D};\ i = j,\sigma);$$

$$I = kD \mid RB\},\ k = j,\sigma$$

$$= \frac{n_k^D - n_k^{'D}}{(n_j^C + n_j^D + n_\sigma^C + n_\sigma^D) - (n_j^{'C} + n_j^{'D} + n_\sigma^{'C} + n_\sigma^{'D})} \cdot PS\left\{\sum_{k_1}\left\{\phi^{(n_j^{'C}+n_\sigma^{'C})}(z_0(t_1)+k_1-e)\right.\right.$$

$$\cdot \left\{\sum_{k_2} \phi(e+k_2)\phi^{(n_j^D+n_\sigma^D-n_j^{'D}-n_\sigma^{'D}-1)}(d_0^D-k_1-k_2-e)\right\} \cdot \phi^{(n_j^{'D}+n_\sigma^{'D})}(k_1)$$

$$\left.\left.\cdot\ \phi^{(n_j^C+n_\sigma^C-n_j^{'C}-n_\sigma^{'C})}(d_0^C-Z_0(t_1)-k_1+e)\right\}\right\} \Bigg/ \left\{\phi^{(n_j^D+n_\sigma^D)}(d_0^D)\right.$$

$$\left.\cdot\ \phi^{(n_j^C+n_\sigma^C)}(d_0^C)\right\},$$

where

$$n_k^{'C} \leq n_k^C\ (\geq 0),\ n_k^{'D} < n_k^D\ (\geq 1);\quad k = j,\sigma;\quad \text{and}$$

$$e = 0,1,\ldots,z_0(t_1) + d_0^D - (n_j^{'C} + n_j^{'D} + n_\sigma^{'C} + n_\sigma^{'D}).$$

The ranges for k_1 and k_2 are,

$$\max\{n_j^{'D}+n_\sigma^{'D},\ n_j^{'C}+n_\sigma^{'C}-z_0(t_1)+e)\} \leq k_1 \leq d_0^C-z_0(t_1)+e-(n_j^C+n_\sigma^C-n_j^{'C}-n_\sigma^{'C}),$$

and

$$1 \leq k_2 \leq d_0^D-k_1-e-(n_j^D+n_\sigma^D-n_j^{'D}-n_\sigma^{'D}).$$

The probability distribution of $U_j(t)$ can now be obtained from Eqs. (5.20 - 5.26). Let

$$PS1 = \frac{1}{(n_j^C+n_j^D+n_\sigma^C+n_\sigma^D) - (n_j^{'C}+n_j^{'D}+n_\sigma^{'C}+n_\sigma^{'D})}$$

Then

$$(5.27)\quad Pr\{U_j(t) = s_j+b\}_B = \sum_{d_0^C > z_0(t_1)} \sum_{d=0}^{s_j+b} Pr\{U_j^1(t) = s_j+b-d\}$$

$$\cdot \left\{ \sum_{n_\sigma^D} \sum_{n_\sigma^C} \sum_{n_j^D} \sum_{n_j^C} \left\{ \sum_{n_\sigma^{'D}} \sum_{n_\sigma^{'C}} \sum_{n_j^{'D}} \sum_{n_j^{'C}} \left(\frac{PS}{PS1}\right) \sum_{d_0^D} \sum_{e} \right.\right.$$

$$\cdot \left\{ (n_\sigma^C - n_\sigma^{'C}) US(\sigma,C) + (n_\sigma^D - n_\sigma^{'D}) US(\sigma,D) + (n_j^C - n_j^{'C}) US(j,C) + (n_j^D - n_j^{'D}) US(j,D) \right\} \Bigg\} \Bigg\}$$

$$\cdot\ P[n_j^C | \lambda_j^C(\tau_0 - R_0)] \cdot P[n_j^D | \lambda_j^D(\tau_0 - R_0)] \cdot P[n_\sigma^C | \lambda_\sigma^C(\tau_0 - R_0)] \cdot P[n_\sigma^D | \lambda_\sigma^D(\tau_0 - R_0)] \Bigg\}$$

where

$$US(\sigma,C) = \sum_{k_1} \Bigg\{ \phi^{(n_j^{'C} + n_\sigma^{'C})}(z_0(t_1) + k_1 - e) \cdot \phi^{(n_j^{'D} + n_\sigma^{'D})}(k_1) \Bigg\{ \sum_{k_2} \phi(e + k_2)$$

$$\cdot \Bigg\{ \sum_{k_3=0}^{d} \phi^{(n_j^C - n_j^{'C})}(k_3) \cdot \phi^{(n_\sigma^C - n_\sigma^{'C} - 1)}(d_0^C - z_0(t_1) - k_1 - k_2 - k_3)$$

$$\cdot\ \phi^{(n_j^D - n_j^{'D})}(d - k_3) \cdot \phi^{(n_\sigma^D - n_\sigma^{'D})}(d_0^D - k_1 - d + k_3) \Bigg\} \Bigg\} \Bigg\},$$

$$US(\sigma,D) = \sum_{k_1} \Bigg\{ \phi^{(n_j^{'C} + n_\sigma^{'C})}(z_0(t_1) + k_1 - e) \cdot \phi^{(n_j^{'D} + n_\sigma^{'D})}(k_1) \Bigg\{ \sum_{k_2} \phi(e + k_2)$$

$$\cdot \Bigg\{ \sum_{k_3=0}^{d} \phi^{(n_j^C - n_j^{'C})}(k_3) \cdot \phi^{(n_\sigma^C - n_\sigma^{'C})}(d_0^C - z_0(t_1) - k_1 - k_3 + e)$$

$$\cdot\ \phi^{(n_j^D - n_j^{'D})}(d - k_3) \cdot \phi^{(n_\sigma^D - n_\sigma^{'D} - 1)}(d_0^D - k_1 - k_2 + e - d + k_3) \Bigg\} \Bigg\} \Bigg\},$$

$$US(j,C) = \sum_{k_1} \left\{ \phi^{(n_j^{'C}+n_\sigma^{'C})}(z_0(t_1)+k_1-e) \cdot \phi^{(n_j^{'D}+n_\sigma^{'D})}(k_1) \left\{ \sum_{k_2=0}^{d} \phi(e+k_2) \right.\right.$$

$$\cdot \left\{ \sum_{k_3} \phi^{(n_j^C-n_j^{'C}-1)}(k_3)\phi^{(n_\sigma^C-n_\sigma^{'C})}(d_0^C-z_0(t_1)-k_1-k_2-k_3) \right.$$

$$\left.\left.\left. \cdot \phi^{(n_j^D-n_j^{'D})}(d-k_2-k_3)\phi^{(n_\sigma^D-n_\sigma^{'D})}(d_0^D-k_1-d+k_2+k_3) \right\}\right\}\right\}$$

and

$$US(j,D) = \sum_{k_1} \left\{ \phi^{(n_j^{'C}+n_\sigma^{'C})}(z_0(t_1)+k_1-e) \cdot \phi^{(n_j^{'D}+n_\sigma^{'D})}(k_1) \left\{ \sum_{k_2=0}^{d} \phi(e+k_2) \right.\right.$$

$$\cdot \left\{ \sum_{k_3} \phi^{(n_j^C-n_j^{'C})}(k_3)\phi^{(n_\sigma^C-n_\sigma^{'C})}(d_0^C-z_0(t_1)-k_1-k_3+e) \right.$$

$$\left.\left.\left. \cdot \phi^{(n_j^D-n_j^{'D}-1)}(d-k_3)\phi^{(n_\sigma^D-n_\sigma^{'D})}(d_0^D-k_1-k_2+e-d+k_2+k_3) \right\}\right\}\right\}.$$

The ranges for the variables in the above enumeration are given in Eqs.(5.25) and (5.26).

Since the right hand sides of Eqs. (5.19) and (5.27) are independent of t, we obtain $Pr\{B_j(*) = b\}$ by substituting these equations and Eq.(5.5) into Eq. (4.3).

The special cases of complete recoverability and complete non-recoverability can be analyzed in a manner similar to that outlined in Section 4.4.1 and 4.4.2, respectively.

5.2.2 Different Order Size Distributions At the Bases

Up to now, all the models studied for the two-echelon system have assumed that order size distributions at the bases are identical In this section, we allow for a more general situation where the order size distributions at the bases are different.

At the depot, compounding distributions for the demand processes $\{D_0(t), t\geq 0\}$, $\{D_0^C(t), t\geq 0\}$ and $\{D_0^D(t), t\geq 0\}$ are given by Eqs. (5.2), (5.3) and (5.4), respectively. Also, Eqs. (5.5), (5.6) and (5.7) provide the stationary distributions for the process $\{Z_0(t), t\geq 0\}$, $\{Q_0(t), t\geq 0\}$ and $\{B_0(t), t\geq 0\}$, respectively.

To obtain the probability distribution of the process $\{B_j(t), t\geq 0\}$, we consider the two cases A and B in the same context as we did in Section 5.2.1. In order to evaluate Eqs. (4.6) and (4.7) representing $Pr\{B_j(t) = b\}$ for the cases A and B, we obtain $Pr\{U_j(t) = s_j+b\}_A$ and $Pr\{U_j(t) = s_j+b\}_B$ in Sections 5.2.2.1 and 5.2.2.2, respectively. To derive these, we use the schemes developed in Sections 5.2.1.1 and 5.2.1.2. Also, we shall use the same notation as used in these Sections.

5.2.2.1 Case (A): $Pr\{U_j(t) = s_j+b\}_A$

To obtain $Pr\{U_j(t) = s_j+b\}_A$, we find the probability distributions for its two components $U_j^1(t)$ and $U_j^2(t)$ as described in Section 5.2.2.1. We derive these by modifying the related expressions of Section 5.2.1.1, for the present case of different order size distributions at the bases.

Substituting $\phi = \phi_j$ in Eq. (5.8), we get

(5.28) $\Pr\{U_j^1(t) = s_j+b-d\} = CP[s_j+b-d\,|\,\lambda_j^B R_j + (\lambda_j^C+\lambda_j^D)\tau_j,\ \phi_j]$.

Furthermore, we see that Eqs. (5.9 - 5.11) describing the conditional probability distribution of $U_j^2(t)$ hold for this case with $\phi = \phi_j$. Following the arguments used in obtaining the expressions for $\Pr\{EX = 0(\geq 0);\ I=i;\ N_j'(t_2,t_3) = n_j';\ N_\sigma'(t_2,t_3) = n_\sigma'|RA\}$ in Eqs. (5.12 - 5.13) we have the following.

(i) $\underline{0 \leq n_j' + n_\sigma' < n_j + n_\sigma;\ (n_j + n_\sigma \geq 1)}$

Here, $\Pr\{Y_1+Y_2+\ldots+Y_{n_j'+n_\sigma'} = z_0(t_1) - d_0^C(t_1,t_2) - e\}$

$$= \sum_{k_1=n_j'}^{z_0(t_1)-d_0^C(t,t_2)-e} \left\{\phi_j^{(n_j')}(k_1)\phi_\sigma^{0(n_\sigma')}(z_0(t_1)-d_0^C(t_1,t_2)-e-k_1)\right\}$$

$$= \phi_0^{(n_j'+n_\sigma')}(z_0(t_1)-d_0^C(t_1,t_2)-e).$$

Then

(5.29) $\Pr\{EX = e;\ I=i;\ N_j'(t_2,t_3) = n_j';\ N_\sigma'(t_2,t_3) = n_\sigma'|RA\}$

$$= \frac{(n_i-n_i')}{(n_j+n_\sigma)-(n_j'+n_\sigma')} \frac{\binom{n_j}{n_j'}\binom{n_\sigma}{n_\sigma'}}{\binom{n_j+n_\sigma}{n_j'+n_\sigma'}} \cdot \left\{\phi_0^{(n_j'+n_\sigma')}(z_0(t_1)-d_0^C(t_1,t_2)-e) \sum_{\ell>e} \phi_i^0(\ell)\right\},$$

for $e = 0,1,\ldots,z_0(t_1)-d_0^C(t_1,t_2)-(n_j'+n_\sigma')$,

$0 \leq n_i' \leq n_i,\quad i = j,\sigma$.

(ii) $\underline{n_j' + n_\sigma' = n_j + n_\sigma \ (\geq 0)}$

$$(5.30)\quad \Pr\{EX = \ ;\ N_j'(t_2,t_3) = n_j';\ N_\sigma'(t_2,t_3) = n_\sigma' \mid RA\}$$

$$\equiv \begin{cases} \displaystyle\sum_{\ell=n_j'+n_\sigma'}^{z_0(t_1)-d_0^C(t_1,t_2)} \phi_0^{(n_j'+n_\sigma')}(\ell), & \text{for } e=0,\ n_j'=n_j,\ n_\sigma'=n_\sigma,\ n_j+n_\sigma \geq 1; \\ 1 & , \text{for } e= z_0(t_1)-d_0^C(t_1,t_2),\ n_j=n_j'=n_\sigma=n_\sigma' = 0; \\ 0 & , \text{otherwise} \end{cases}$$

We can now obtain $\Pr\{U_j^2(t) = 0 \mid RA\}$ and $\Pr\{U_j^2(t) = d \mid RA\}$, $d \geq 1$.

Thus we have

$$(5.31)\quad \Pr\{U_j(t)=s_j+b\}_A = \sum_{d_0^C(t_1,t_2)=0}^{z_0(t_1)} \Bigg[CP[s_j+b \mid \lambda_j^B R_j + \lambda_j^0 \tau_j, \phi_j] \cdot \Bigg\{ \sum_{n_j+n_\sigma=0}^{z_0(t_1)-d_0^C(t_1,t_2)} \left(\sum_{\ell=n_j+n_\sigma}^{z_0(t_1)-d_0^C(t_1,t_2)} \phi_0^{(n_j+n_\sigma)}(\ell) \right)$$

$$\cdot P[n_j+n_\sigma \mid \lambda_0 R_0] \quad + \Bigg\{ \sum_{n_j=0}^{z_0(t_1)-d_0^C(t_1,t_2)} \sum_{n_\sigma=1}^{\infty} \left\{ \sum_{n_\sigma'=0}^{n_\sigma-1} \frac{n_\sigma!(n_j+n_\sigma')!}{n_\sigma'!(n_j+n_\sigma)!} \sum_e \phi_0^{(n_j+n_\sigma')}(z_0(t_1)-d_0^C(t_1,t_2)-e) \sum_{\ell>e} \phi_\sigma^0(\ell) \right\}$$

$$\cdot P[n_\sigma \mid \lambda_\sigma^0 R_0] \cdot P[n_j \mid \lambda_j^0 R_0] \Bigg\}\Bigg\} \qquad + $$

$$\sum_{d=1}^{s_j+b} CP[s_j+b-d \mid \lambda_j^B R_j + \lambda_j^0 \tau_j, \phi_j] \cdot \Bigg\{ \sum_{n_j=1}^{\infty} \sum_{n_\sigma=0}^{\infty} \Bigg\{ \sum_{n_j'=0}^{n_j-1} \sum_{n_\sigma'=0}^{n_\sigma} \frac{\binom{n_j}{n_j'}\binom{n_\sigma}{n_\sigma'}}{\binom{n_j+n_\sigma}{n_j'+n_\sigma'}} \cdot \Bigg\{ \sum_e \phi_0^{(n_j'+n_\sigma')}(z_0(t_1) - d_0^C(t_1,t_2)-e)$$

$$\left\{ \frac{n_j-n_j'}{n_j+n_\sigma-n_j'-n_\sigma'} \left(\sum_{k>0}^{d} \phi_j(k+e) \cdot \phi_j^{(n_j-n_j'-1)}(d-k) \right) + \frac{n_\sigma-n_\sigma'}{n_j+n_\sigma-n_j'-n_\sigma'} \left(\phi_j^{(n_j-n_j')}(d) \sum_{\ell>e} \phi_\sigma^0(\ell) \right) \right\}\Bigg\}\Bigg\}\Bigg\} P[n_\sigma \mid \lambda_\sigma^0 R_0] \cdot P[n_j \mid \lambda_j^0 R_0]$$

$$\cdot\ CP[d_0^C(t_1,t_2) \mid \lambda_0^C(\tau_0-R_0), \phi_0^C].$$

where $0 \le e \le z_0(t_1) - d_0^C(t_1) - (n_j' + n_\sigma')$.

5.2.2.2 Case B: $Pr\{U_j(t) = s_j+b\}_B$

Here again, the probability distributions of $U_j^1(t)$ and $U_j^2(t)$ for case B are derived using the notation and the method described in Section 5.2.1.2.

Substituting ϕ_j for ϕ into Eq. (5.20), we have

$$Pr\{U_j^1(t) = s_j+b-d\} = CP[s_j+b-d \mid \lambda_j^B R_j + \lambda_j^0(\tau_j + R_0),\ \phi_j].$$

Similarly, substituting ϕ_0^C and ϕ_0^D in Eq. (5.21), we get

$$Pr\{RB2 \mid RB1\} = \phi_0^{C^{(n_j^C+n_\sigma^C)}}(d_0^C) \cdot \phi_0^{D^{(n_j^D+n_\sigma^D)}}(d_0^D).$$

Also, Eqs. (5.22 - 5.24) hold for this case with $\phi(\cdot) = \phi_j(\cdot)$. Following the steps used in deriving Eq. (5.25) we have

$$Pr\{EX = e;\ (N_i^{'C}(t_1,t_2) = n_i^{'C};\ N_i^{'D}(t_1,t_2) = n_i^{'D};\ i=j,\sigma);\ I=kC \mid RB\},\ k=j,\sigma$$

$$= \frac{(n_k^C - n_k^{'C})}{(n_j^C + n_j^D + n_\sigma^C + n_\sigma^D) - (n_j^{'C} + n_j^{'D} + n_\sigma^{'C} + n_\sigma^{'D})} \cdot PS$$

$$\cdot \left\{ \sum_{k_1} \left\{ \phi_0^{C^{(n_j^{'C}+n_\sigma^{'C})}}(z_0(t_1) + k_1 - e) \right.\right.$$

$$\left.\left. \cdot \left\{ \sum_{k_2} \phi_k^C(e+k_2)\phi_0^{C^{(n_j^C+n_\sigma^C-n_j^{'C}-n_\sigma^{'C}-1)}}(d_0^C - z_0(t_1) - k_1 - k_2) \right\}\right\}\right.$$

$$\cdot \left(\phi_0^{D(n_j^{'D}+n_\sigma^{'D})}(k_1) \cdot \phi_0^{D}{}^{(n_j^D+n_\sigma^D-n_j^{'D}-n_\sigma^{'D})}(d_0^D-k_1)\right) \Big/ \left\{\phi_0^{C}{}^{(n_j^C+n_\sigma^C)}(d_0^C)\,\phi_0^{D(n_j^D+n_\sigma^D)}(d_0^D)\right\}, \quad k = j,\sigma ,$$

where the ranges for k_1 and k_2 are the same as given in Eq. (5.25). Similarly, we can obtain $Pr\{EX = e;\ (N_i^{'C}(t_1,t_2) = n_i^{'C};\ N_i^{'D}(t_1,t_2) = n_i^{'D};\ i = j,\sigma);\ I = kD|RB\}$, $k = j,\sigma$. Then proceeding in a manner similar to that in deriving Eq. (5.27), we can obtain $Pr\{U_j(t)=s_j+b\}_B$. This involves a substantial amount of enumeration. The detailed expression is omitted here.

The special cases of complete recoverability and complete non-recoverability can be dealt in the same way as mentioned in Section 4.4. We emphasize that approach developed in this section can also be used for a more general case where the three demand processes $\{D_j^B(t),\ t\geq 0\}$, $\{D_j^C(t),\ t\geq 0\}$ and $\{D_j^D(t),\ t\geq 0\}$ at base j $(j = 1,2,\ldots,J)$ are independent compound Poisson processes with different compounding distributions.

5.3 The Unit Model

In this model, upon arrival of a requisition at a base, each unit in the failed batch is inspected independently. There are three possible outcomes of each inspection. At base j, a unit is base repairable with probability r_j, is depot repairable with probability $(1-r_j)\rho$, or with probability $(1-r_j)(1-\rho)$ is condemned.

We shall use the following nomenclature in addition to those introduced in Sections 4.1 and 5.1.

D_j = the number of units in a batch failed at base j $(j=1,2,\ldots,J)$.

D_j^B = the number of base repairable units in a batch failed at base j $(j=1,2,\ldots,J)$.

D_j^C = the number of condemned units in a batch failed at base j $(j=1,2,\ldots,J)$.

D_j^D = the number of depot repairable units in a batch failed at base j $(j=1,2,\ldots,J)$.

D_j^0 = the number of units demanded from depot upon a failure at base j $(j=1,2,\ldots,J)$.

Obviously, $D_j^0 = D_j^C + D_j^D$.

As an implication of inspection under the unit model, we have

$$(5.32)\quad \Pr\{D_j^B = d_j^B;\ D_j^C = d_j^C;\ D_j^D = d_j^D \mid D_j = d_j\}$$

$$= \begin{cases} \dfrac{d_j!}{d_j^B!\,d_j^C!\,d_j^D!}\,[r_j]^{d_j^B}\,[(1-r_j)(1-\rho)]^{d_j^C}[(1-r_j)\rho]^{d_j^D}, \\ \qquad\qquad \text{for } d_j^B, d_j^C, d_j^D \geq 0, \text{ and } d_j = d_j^B + d_j^C + d_j^D; \\ 0 \qquad\qquad\qquad , \text{ otherwise.} \end{cases}$$

It can be easily shown from Eq. (5.32) that the conditional probabilities of D_j^B, D_j^C and D_j^D given $D_j = d_j$ are $B[d_j; r_j]$, $B[d_j; (1-r_j)(1-\rho)]$ and $B[d_j; (1-r_j)\rho]$, respectively; where $B[n; p]$ denotes the binomial probability distribution with parameters n and p (n=number of trials, p = probability of success, $0 < p < 1$). Therefore

$$\phi_j^B(k) = \Pr\{D_j^B = k\} = \sum_{d_j=k}^{\infty} \binom{d_j}{k} [r_j]^k [1-r_j]^{d_j-k} \phi_j(d_j) ,$$

$$\phi_j^C(k) = \Pr\{D_j^C = k\} = \sum_{d_j=k}^{\infty} \binom{d_j}{k} [(1-r_j)(1-\rho)]^k [r_j+(1-r_j)\rho]^{d_j-k} \phi_j(d_j),$$

and

$$\phi_j^D(k) = \Pr\{D_j^D = k\} = \sum_{d_j=k}^{\infty} \binom{d_j}{k} [(1-r_j)\rho]^k [1-(1-r_j)\rho]^{d_j-k} \phi_j(d_j).$$

Furthermore, let $\phi_j(k_1,k_2) = \Pr\{D_j^C = k_1;\ D_j^D = k_2\}$. Then it follows from Eq. (5.32) that

$$(5.33)\quad \phi_j(k_1,k_2) = \sum_{d_j=k_1+k_2}^{\infty} \frac{d_j!}{k_1!k_2!(d_j-k_1-k_2)!}$$

$$\cdot\ [(1-r_j)(1-\rho)]^{k_1}[(1-r_j)\rho]^{k_2}[r_j]^{d_j-k_1-k_2}\ \phi_j(d_j),$$

$$k_1,k_2 \geq 0,$$

and, therefore,

$$(5.34)\quad \phi_j^0(k) = \Pr\{D_j^C + D_j^D = k\} = \sum_{k_1=0}^{k} \phi_j(k_1,\ k-k_1)$$

$$= \sum_{d_j=k}^{\infty} \binom{d_j}{k} (1-r_j)^k r_j^{d_j-k}\ \phi_j(d_j), \qquad k \geq 0.$$

We note that in this case D_j^0 is a non-negative variable, whereas in the case of batch model it is strictly positive. In our analysis, we shall also include the base requisitions levied on the depot for which the order size is zero. Consequently, the depot demand process $\{D_0(t),\ t\geq 0\}$ is a compound Poisson process with parameter $\lambda_0 = \sum_{j=1}^{J} \lambda_j$ and compounding distribution $\phi_0(k) = \frac{1}{\lambda_0} \sum_{j=1}^{J} \lambda_j \phi_j^0,\ k\geq 0$. For $n\geq 1$, let

$$\phi_j^{(n)}(k_1,k_2) = \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} \phi_j^{(n-1)}(k_1-j_1,\ k_2-j_2)\phi_j(j_1,j_2),\quad j_1,j_2 \geq 0,$$

where $\phi_j^{(0)}(0,0) = 1$ and $\phi^0(k_1,k_2) = 0$ for $k_1,k_2 \geq 1$. Then the process $\{D_0^C(t),\ D_0^D(t),\ t\geq 0\}$ is a compound Poisson process with parameter λ_0 and compounding distribution $\phi_0(k_1,k_2) = \frac{1}{\lambda_0} \sum_{j=1}^{J} \lambda_j\phi_j(k_1,k_2)$; that is,

$$(5.35)\quad \Pr\{D_0^C(t) = d_0^C;\ D_0^D(t) = d_0^D\} = \sum_{n=0}^{\infty} \phi_0^{(n)}(d_0^C,d_0^D)\ \frac{e^{-\lambda_0 t}(\lambda_0 t)^n}{n!}$$

$$d_0^C,\ d_0^D \geq 0.$$

In addition, the processes $\{D_0^C(t),\ t\geq 0\}$ and $\{D_0^D(t),\ t\geq 0\}$ are compound Poisson processes with common parameter λ_0 and compounding distributions $\phi_0^C(k) = \frac{1}{\lambda_0} \sum_{j=1}^{J} \lambda_j \phi_j^C(k),\ k\geq 0$, and $\phi_0^D(k) = \frac{1}{\lambda_0} \sum_{j=1}^{J} \lambda_j\ \phi_j^D(k)$, $k\geq 0$, respectively. The depot can now be analyzed as a single location system where recoverable and non-recoverable demand processes are dependent compound Poisson processes. Therefore, the results derived in Section 3.4.2 apply.

From Theorem 3.5, upon substituting $\phi_0^C(\cdot)$ for $p(\cdot)$, we have

$$(5.36)\quad \lim_{t\to\infty} \Pr\{Z_0(t) = k \mid Z(0) = i,\ i \geq p_0\} = \begin{cases} \dfrac{g(S-k)}{1+G(S-s-1)}, & k=s+1,\ldots,S-1, \\ \dfrac{1}{1+G(S-s-1)} & k=S, \end{cases}$$

where $g(1) = \phi_0^C(1)/[1-\phi_0^C(0)]$, $g(k) = [\phi_0^C(k) + \sum_{q=1}^{k-1} \phi_0^C(q)g(k-q)]/[1-\phi_0^C(0)]$, $k=2,3,\ldots$, and $G(k)\ \sum_{\ell=1}^{k} g(\ell)$.

Also, it easily follows from the previous discussions that

$$(5.37)\quad \lim_{t\to\infty} \Pr\{Q_0(t) = q_0 \mid Q_0(0) = 0\} = CP[q_0 \mid \lambda_0 R_0,\ \phi_0^D],\ q_0 \geq 0.$$

The transient distribution of the process $\{B_0(t),\ t \geq 0\}$ can be obtained from Eq. (3.43). As mentioned in Section 3.3.3, the derivation of stationary distribution is computationally complex. The results, however, can be obtained using Laplace transforms. The explicit derivation is omitted here.

To obtain the stationary distribution of the process $\{B_j(t),\ t \geq 0\}$, we assume that the base repair time does not exceed the base procurement lead time; that is, $R_j \geq \tau_j$. The analogous results for the case $R_j > \tau_j$ can be derived using the approach described here.

Here again, we compute $\Pr\{U_j(t) = s_j+b\}$ for the two cases A and B described in Section 4.2.

5.3.1 Case A: $\Pr\{U_j(t) = s_j+b\}_A$

To obtain $\Pr\{U_j(t) = s_j+b\}_A$, we find the probability distributions for $U_j^1(t)$ and $U_j^2(t)$, the two components of $U_j(t)$ in case A. Under the assumption $R_j \leq \tau_j$, it is clear that

$$U_j^1(t) = D_j^C(t_3,\ t-R_j) + D_j^D(t_3,\ t-R_j) + D_j(t-R_j,\ t)$$

$$= D_j^0(t_3,\ t-R_j) + D_j(t-R_j,\ t).$$

Because the arrival process is Poisson, $D_j^0(t_3,\ t-R_j)$ and $D_j(t-R_j,\ t)$ are independent. Therefore

$$\text{(5.38)}\quad \Pr\{U_j^1(t) = s_j+b-d\} = \sum_{k=0}^{s_j+b-d} \left\{ CP[s_j+b-d-k \mid \lambda_j(\tau_j-R_j),\ \phi_j^0] \cdot CP[k \mid \lambda_j R_j, \phi_j] \right\}$$

To obtain $\Pr\{U_j^2(t) = d\}$, we proceed as in section 5.2.2.1. Note that D_j^0 is now a non-negative variable. Then from Eq. (5.31), it follows that

$$
\begin{aligned}
(5.39)\quad & \Pr\{U_j(t)=s_j+b\}_A = \sum_{d_0^C(t_1,t_2)=0}^{z_0(t_1)} \Bigg[\Pr\{U_j^1(t)=s_j+b\} \cdot \Bigg\{ \sum_{n_j+n_\sigma=0}^{\infty} \Bigg(\sum_{\ell=0}^{z_0(t_1)-d_0^C(t_1,t_2)} \varphi_\sigma^{(n_j+n_\sigma)}(\ell) \Bigg) \\
& \cdot P[n_j+n_\sigma|\lambda_0 R_0] + \Bigg\{ \sum_{n_j=0}^{\infty} \sum_{n_\sigma=1}^{\infty} \Bigg\{ \sum_{n'_\sigma=0}^{n_\sigma-1} \frac{n'_\sigma!(n_j+n'_\sigma)!}{n_\sigma!(n_j+n_\sigma)!} \sum_{e=0}^{z_0(t_1)-d_0^C(t_1,t_2)} \Bigg(\sum_{k=0}^{z_0(t_1)-d_0^C(t_1,t_2)-e} \phi_j^{0\,(n_j)}(k) \\
& \cdot \phi_\sigma^{0\,(n'_\sigma)}(z_0(t_1)-d_0^C(t_1,t_2)-e-k) \Bigg) \cdot \sum_{\ell>e} \phi_\sigma^0(\ell) \Bigg\} \cdot P[n_\sigma|\lambda_\sigma R_0] \cdot P[n_j|\lambda_j R_0] \Bigg\} \Bigg\} \quad + \\
& \sum_{d=1}^{s_j+b} \Pr\{U_j^1(t)=s_j+b-d\} \cdot \Bigg\{ \sum_{n_j=1}^{\infty} \sum_{n_\sigma=0}^{\infty} \Bigg\{ \sum_{n'_j=0}^{n_j-1} \sum_{n'_\sigma=0}^{n_\sigma} \frac{\binom{n_j}{n'_j}\binom{n_\sigma}{n'_\sigma}}{\binom{n_j+n_\sigma}{n'_j+n'_\sigma}} \Bigg\{ \sum_{e=0}^{z_0(t_1)-d_0^C(t_1,t_2)} \sum_{k_1=0}^{z_0(t_1)-d_0^C(t_1,t_2)-e} \\
& \cdot \phi_j^{0\,(n'_j)}(k_1) \cdot \varphi_\sigma^{0\,(n'_\sigma)}(z_0(t_1)-d_0^C(t_1,t_2)-e-k_1) \Bigg\{ \frac{n_j-n'_j}{n_j+n_\sigma-n'_j-n'_\sigma} \Bigg(\sum_{k>0}^{d} \phi_j^0(k+e)\, \varphi_j^{0\,(n_j-n'_j-1)}(d-k) \\
& + \frac{n_\sigma-n'_\sigma}{n_j+n_\sigma-n'_j-n'_\sigma} \Bigg(\phi_\sigma^{0\,(n_\sigma-n'_\sigma)}(d) \sum_{\ell>e} \phi_\sigma^0(\ell) \Bigg) \Bigg) \Bigg\} \Bigg\} \Bigg\} \; P[n_\sigma|\lambda_\sigma^0 R_0] \cdot P[n_j|\lambda_j R_0] \cdot CP[d_0^C(t_1,t_2)|\lambda_0(\tau_0-R_0),\, \varphi_0^C].
\end{aligned}
$$

where $\lambda_0 = \sum_{j=1}^{J} \lambda_j$, $\lambda_\sigma = \sum_{\substack{j=1 \\ j\neq i}}^{J} \lambda_j$ and $\phi_\sigma^0 = \frac{1}{\lambda_\sigma} \sum_{\substack{i=1 \\ i\neq j}}^{J} \lambda_i \phi_j^0$. The probability distribution of $U_j^1(t)$ is obtained from Eq. (5.38).

5.3.2 Case (B): $\Pr\{U_j(t) = s_j+b\}_B$

Here again, we obtain the probability distributions for $U_j^1(t)$ and $U_j^2(t)$, the two components of $U_j(t)$ in case B. By definition, as given in Section 4.2

$$U_j^1(t) = D_j^C(t_2,\ t-R_j) + D_j^D(t_2,\ t-R_j) + D_j^B(t-R_j,\ t).$$

Hence,

$$(5.40)\quad \Pr\{U_j^1(t) = s_j+b-d\} = \sum_{k=0}^{s_j+b-d} CP[s_j+b-d-k \mid \lambda_j(\tau_j+R_0-R_j),\ \phi_j^0] \cdot CP[k \mid \lambda_j R_j,\ \phi_j].$$

As indicated earlier, in the unit model each requisition at the depot is associated with a demand for replacement for some depot repairable units and some condemned units. To obtain the probability distribution of $U_j^2(t)$, we follow the steps described in Section 5.2.2.2. Let

$RB1 \equiv \{N_j(t_2,t_3) = n_j;\ N_\sigma(t_2,t_3) = n_\sigma;\ Z_0(t_1) = z_0(t_1)\}$,

$RB2 \equiv \{D_0^C(t_1,t_2) = d_0^C;\ D_0^D(t_1,t_2) = d_0^D\}$

and

$RB = RB1 \cup RB2$

Then similar to Eqs. (5.9), (5.10) and (5.11), we have

(5.41) $\Pr\{U_j^2(t) = d \mid RB;\ EX = 0;\ N_j'(t_1,t_2) = n_j';\ N_\sigma'(t_1,t_2) = n_\sigma'\}$

$$= \phi_j^{0\ (n_j - n_j')}(d),$$

(5.42) $\Pr\{U_j^2(t) = d \mid RB;\ EX = e > 0;\ N_j'(t_1,t_2) = n_j';\ N_\sigma'(t_1,t_2) = n_\sigma',\ I = \sigma\}$

$$= \phi_j^{0\ (n_j - n_j')}(d)$$

and

(5.43) $\Pr\{U_j^2(t) = d \mid RB;\ EX = e > 0;\ N_j'(t_1,t_2) = n_j';\ N_\sigma'(t_1,t_2) = n_\sigma';\ I = j\}$

$$= \sum_{k>0}^{d} \phi_j^0(k+e)\ \phi_j^{0(n_j - n_j' - 1)}(d-k).$$

Following the arguments used in deriving Eqs. (5.25) and (5.26), we obtain

(5.44) $\Pr\{EX = e;\ N_j'(t_1,t_2) = n_j';\ N_\sigma'(t_1,t_2) = n_\sigma',\ I = k \mid RB\},\ k = j,\sigma$

$= \Pr\{(Y_1+Y_2+\ldots+Y_{n_j'}) + (Y_1+Y_2+\ldots+Y_{n_\sigma'}) = z_0(t_1) + d_0^D - e,$

$Y_{n_k'+1} > e$; out of first $(n_j'+n_\sigma')$ requistions at the

depot, n_j' are from source j; the next demand is

from source $k \mid D_0^C(t_1,t_2) = d_0^C;\ D_0^D(t_1,t_2) = d_0^D;\ RB1\}$;

$k = j,\sigma$

$$= \frac{n_k - n_k'}{(n_j+n_\sigma) - (n_j'+n_\sigma')}\ \frac{\binom{n_j}{n_j'}\binom{n_\sigma}{n_\sigma'}}{\binom{n_j + n_\sigma}{n_j' + n_\sigma'}} \left\{ \sum_{d_1} \phi_0^{(n_j'+n_\sigma')}(d_1,\ z_0(t_1) + d_0^D - e - d_1) \right.$$

$$\cdot\left\{\sum_{k_2}\sum_{d_2}\phi_k^0(e+k_2-d_2,\ d_2)\right.$$

$$\left.\left.\cdot\ \phi_0^{(n_j+n\ -n_j'-n_\sigma'-1)}(d_0^C-d_1-e-k_2+d_2,d_1+e-d_2-z_0(t_1)\right\}\right\}\Bigg/$$

$$\phi_0^{(n_j+n_\sigma)}(d_0^C,d_0^D);$$

where,

$$n_k' \leq n_k(\geq 1),\ k = j,\sigma;\ \text{and}\quad e = 0,1,\ldots,z_0(t_1) + d_0^D.$$

From the property of $\phi^n(a,b)$, we see that the ranges for the indicics in the summations must satisfy

$$0 \leq d_1 \leq z_0(t_1) + d_0^D - e,\quad 1 \leq k_2 \leq d_0^C - d_1 - e \quad \text{and} \quad 0 \leq d_2 \leq d_1 + e - z_0(t_1).$$

Thus from Eqs. (5.41 - 5.44) we get

$$(5.45)\quad \Pr\{U_j(t) = s_j+b\}_B = \sum_{d_0^C > z_0(t_1)}\ \sum_{d=0}^{s_j+b} \Pr\{U_j^1(t) = s_j+b-d\}$$

$$\cdot\left\{\sum_{n_j}\sum_{n_\sigma}\left\{\sum_{n_j'}\sum_{n_\sigma'}\frac{\binom{n_j}{n_j'}\binom{n_\sigma}{n_\sigma'}}{\binom{n_j+n_\sigma}{n_j'+n_\sigma'}}\ \frac{1}{(n_j+n_\sigma)-(n_j'+n_\sigma')}\sum_{d_0^D=0}^{\infty}\sum_{e}\right.\right.$$

$$\left.\cdot\left\{(n_\sigma-n_\sigma')\cdot US(\sigma) + (n_j-n_j')\cdot US(j)\right\}\right\}$$

$$\left.\cdot\ P[n_j|\lambda_j(\tau_0-R_0)]\cdot P[n_\sigma|\lambda_\sigma(\tau_0-R_0)]\right\},$$

where

$$US(\sigma) = \sum_{d_1} \phi_0^{(n_j'+n_\sigma')}(d_1,\ z_0(t_1) + d_0^D - e - d_1) \sum_{k_2} \sum_{d_2} \phi_\sigma^0(e+k_2-d_2,\ d_2)$$

$$\cdot \left\{ \sum_{k_3=0}^{d} \phi_j^{0\ (n_j + n_j')}(d-k_3,\ k_3) \cdot \phi_\sigma^{0\ (n_\sigma - n_\sigma' - 1)}(d_0^C - d_1 - e - k_2 + d_2 - d + k_3,\ d_1 + e - d_2 - z_0(t_1) - k_3) \right\},$$

and

$$US(j) = \sum_{d_1} \phi_0^{(n_j'+n_\sigma')}(d_1,\ z_0(t_1) + d_0^D - e - d_1) \sum_{k_2>0}^{d} \sum_{d_2} \phi_j^0(e+k_2-d_2,\ d_2)$$

$$\cdot \left\{ \sum_{k_3} \phi_j^{0\ (n_j - n_j' - 1)}(d-k_2-k_3,\ k_3) \cdot \phi_\sigma^{0\ (n_\sigma - n_\sigma')}(d_0^C - d_1 - e + d_2 - d + k_3,\ d_1 + e - d_2 - z_0(t_1) - k_3) \right\}$$

The ranges for the indicies in the summations can be easily obtained using the fact that in an expressinn $\phi^n(a,b)$, a, $b \geq 0$ for $n \geq 1$, and $a = b = 0$ for $n = 0$.

Thus $Pr\{B_j(*) = b\}$ can now be obtained from Eqs. (5.36), (5.39) and (5.45).

CHAPTER VI

A COMPARISON OF TWO-ECHELON MODELS

In this chapter we compare the two-echelon inventory models studied in the previous two chapters to Sherbrooke's METRIC (Multi-Echelon Technique for Recoverable Item Control) model [17]. In Section 6.1, we briefly describe the METRIC model. In Section 6.2, a general comparison is presented, while a computational comparison is made in Section 6.3.

6.1 The METRIC Model

Almost a decade ago Sherbrooke [17] developed the well-known METRIC model for a two-echelon system similar to that described in section 1.1.2. A major purpose of the model is to determine optimal base and depot stock levels that minimize the expected number of total system backorders subject to a constraint on system investment or system performance.

Demand at base j is assumed to be represented by a compound Poisson process with parameter λ_j $(j=1,2,\ldots,J)$. Upon arrival of a requisition, one or several units are demanded for replacement and a like number of failed units are turned in for repair at the base. A batch of failed units at base j is repaired at the base with probability r_j and is shipped to the depot for repair with probability $(1-r_j)$. Thus there are no condemnations; that is, the system is by definition conservative. Bases use an $(s-1, s)$ policy for procurement of serviceable units from the depot. Repair time at location j is a random variable with finite and known mean R_j $(j=0,1,\ldots,J)$.

It is assumed that the repair time is the same for each unit in a failed batch. The order-and-ship time at base j is also a random variable with mean $\tau_j (j=1,2,\ldots,J)$. Furthermore, it is assumed that the order size distribution is the same at all bases. Let $\bar{\phi}$ be the mean order size per requisition at the bases. Then the mean depot demand rate is $\sum_{j=1}^{J} \lambda_j(1-r_j)\bar{\phi} = \lambda_0\bar{\phi}$, where $\lambda_0 = \sum_{j=1} (1-r_j)\lambda_j$.

In the METRIC model, the stationary distribution of the number of backorders at a base is derived from Eq. (12) of Feeney and Sherbrooke [5]. Let S be the spare stock (inventory on hand + on order + in repair - backorders) for an item where demands are compound Poisson with parameter λ, and resupply time is a random variable with mean T. Also, assume that the resupply time is the same for all units demanded by a requisition. Then in the case where backlogging is allowed, the number of units in resupply has a Poisson distribution with mean λT.

Let S_0 be the spare stock at the depot and $T_j(S_0)$ be the average response time to a demand from base j. Sherbrooke shows that $T_j(S_0)$ can be expressed as

$$(6.1) \qquad T_j(S_0) = r_j R_j + (1-r_j)(\tau_j + \delta(S_0)\cdot R_0)$$

where

$\delta(S_0)\cdot R_0$ = average delay per depot demand

= expected number of backorders at the depot/average depot demand rate

$$= \left(\sum_{x=S_0+1}^{\infty} (x-S_0)P[x|\lambda_0 R_0]\right) \Big/ \lambda_0\bar{\phi},$$

where $P[\cdot|m]$ denotes the Poisson distribution with mean m. Thus in the METRIC model, the stationary distribution of the number of backorders at base j is given by

$$(6.2) \qquad \Pr\{B_j(*) = b\} = P[s_j+b|\lambda_j T_j(S_0)]$$

$$= P[s_j+b|\lambda_j^B R_j+\lambda_j^0 \tau_j+\lambda_j^0 \ \delta(S_0)\cdot R_0]$$

$$b=-s_j,-s_j+1,\ldots,0,1,\ldots;$$

where $\lambda_j^B = r_j\lambda_j$ and $\lambda_j^0 = (1-r_j)\lambda_j$.

6.2 General Comparison

We now compare some of the features of the METRIC model and the two-echelon models studied in chapter IV and V. Our models are more general than METRIC in that they permit non-recoverability and positive condemnation rates. Our analysis includes both batch and unit models for inspection of failed units, whereas METRIC considers only the batch model. We also examined the case where order size distributions are different at the base. The METRIC model is confined to the case where the order size distribution is the same at all bases.

The METRIC model, on the other hand, is more general in that it allows random repair and order-and-ship times. Furthermore, METRIC provides simple but approximate expressions for the stationary distribution of the number of backorders outstanding at a location at any point in time. In our analysis, the expressions are relatively more complex and a large amount of enumeration is required to compute the probability distributions.

6.3 A Computational Comparison

In this section, we assess the degree to which the METRIC model could serve as a useful approximation to our model which we shall refer to as the EXACT model. Also, we compare the computational complexities of the two models in terms of their execution times on a digital computer.

We consider the case of *unit demands at the bases.* For the EXACT model, the conservative system is discussed in section 4.4.1 and the stationary distribution of the number of backorders at the bases is given by Eq. (4.25). For the METRIC model it is computed from Eqs. (6.1) and (6.2) upon substituting $\bar{\phi} = 1$ into Eq. (6.1). Furthermore, the term $\delta(S_0)\cdot R_0$ is simplified as a finite sum and the resulting expression is given by

$$(6.3)\quad \delta(S_0)\cdot R_0 = \frac{1}{\lambda_0}[\lambda_0 \cdot \{1 - \sum_{x=0}^{S_0-1} P[x|\lambda_0 R_0]\} - S_0 \cdot \{1 - \sum_{x=0}^{S} P[x|\lambda_0 R_0]\}]$$

For the purpose of computational comparison, we considered several numerical examples covering a wide range of the system parameters - demand rates at bases, depot and base repair times, depot and base spare stock levels and probability of a unit being repairable at a base. When the comparison was made, the difference between the METRIC model and the EXACT model was found to have the same pattern in all cases. The extent of discrepancy between the two models, however, did vary from one example to the other depending on the numerical values of their parameters. To describe the comparison between the two models, the following example was selected as representative of all situations considered.

<u>Example</u> Number of bases = 5

Base (j)	Arrival rate (per day) (λ_j)	Order-and-ship time (days) (τ_j)	Repair time (days) (R_j)
1	0.07098	12	6
2	0.14766	12	6
3	0.01497	12	6
4	0.04591	12	6
5	0.11019	12	6

Repair time at the depot (R_0) = 56 days

The stationary distribution for the number of units in resupply at the bases is computed for the two models. Tables I, II and III show such distributions for $r_j = 0.10$, $r_j = 0.50$ and $r_j = 0.90$, respectively. It is assumed that r_j is the same for all bases. The corresponding arrival rate at the depot (λ_0) is stated at the beginning of each table. The depot spare stock level (S_0) for each case is chosen to be approximately equal to $\lambda_0 R_0$, the average number of units in resupply at the depot. Similarly the value of s_j, the spare stock level at base j, is chosen to be the least integer greater than or equal to $\lambda_j[(1-r_j)\tau_j + r_j R_j]$. The letter x refers to the number of units in resupply. The expected number of backorders (EBO) is computed in the last row of each table. The execution time for the two models is stated below each table. All programs were written in FORTRAN IV and run on Cornell's IBM 370/168 computer.

The data displayed in the tables indicated that the METRIC model

Table I

The Probability Distribution of the Number of Units in Resupply at the Bases

$r_j = 0.10,\ j = 1,\ldots,5$

$s_1 = 1,\ s_2 = 2,\ s_3 = 1,\ s_4 = 1,\ s_5 = 2$

$\lambda_0 = 0.35074,\ S_0 = 19$

x	Base 1		Base 2		Base 3		Base 4		Base 5	
	METRIC	EXACT	METRIC	EXACT	METRIC	EXACT	METRIC	EXACT	METRIC	EXACT
0	0.304245	0.332759	0.084139	0.116187	0.778089	0.781654	0.463163	0.482003	0.157674	0.191866
1	0.362028	0.341539	0.208268	0.225341	0.195234	0.189111	0.356486	0.334729	0.291260	0.289041
2	0.215392	0.195209	0.257762	0.233829	0.024494	0.026191	0.137189	0.131208	0.269011	0.237308
3	0.085433	0.084543	0.212677	0.177541	0.002049	0.002774	0.035197	0.039235	0.165641	0.145229
4	0.025415	0.031272	0.131609	0.113358	0.000129	0.000248	0.006773	0.009995	0.076494	0.075602
5	0.006048	0.010372	0.065154	0.065429	0.000006	0.000019	0.001043	0.002264	0.028260	0.035576
6	0.001199	0.003140	0.026879	0.035254	0.000000	0.000001	0.000134	0.000464	0.008701	0.015486
7	0.000204	0.000874	0.009505	0.017901	0.000000	0.000000	0.000015	0.000087	0.002296	0.006276
8	0.000030	0.000225	0.002941	0.008573	0.000000	0.000000	0.000001	0.000015	0.000530	0.002374
9	0.000004	0.000054	0.000809	0.003872	0.000000	0.000000	0.000000	0.000002	0.000109	0.000840
10	0.000000	0.000012	0.000200	0.001652	0.000000	0.000000	0.000000	0.000000	0.000020	0.000279
11	0.000000	0.000003	0.000045	0.000667	0.000000	0.000000	0.000000	0.000000	0.000003	0.000087
12	0.000000	0.000000	0.000009	0.000255	0.000000	0.000000	0.000000	0.000000	0.000001	0.000026
13	0.000000	0.000000	0.000002	0.000093	0.000000	0.000000	0.000000	0.000000	0.000000	0.000007
14	0.000000	0.000000	0.000000	0.000032	0.000000	0.000000	0.000000	0.000000	0.000000	0.000002
15	0.000000	0.000000	0.000000	0.000011	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
EBO	0.414166	0.522680	0.851829	0.932996	0.029003	0.032569	0.232840	0.251680	0.453833	0.519999

Execution time: METRIC = 0.08 sec.

EXACT = 1.97 sec.

Table II

The Probability Distribution of the Number of Units in Resupply at the Bases

$r_j = 0.50,\ j = 1,\ldots,5$

$s_1 = 1,,\ s_2 = 1,\ s_3 = 1,\ s_4 = 1,\ s_5 = 1$

$\lambda_0 = 0.19486,\ S_0 = 11$

x	Base 1		Base 2		Base 3		Base 4		Base 5	
	METRIC	EXACT	METRIC	EXACT	METRIC	EXACT	METRIC	EXACT	METRIC	EXACT
0	0.419182	0.436670	0.163877	0.191240	0.832486	0.834161	0.569847	0.580133	0.259310	0.284252
1	0.364458	0.346243	0.296394	0.294385	0.152627	0.149626	0.320475	0.306360	0.349999	0.336554
2	0.158439	0.150727	0.268035	0.242343	0.013991	0.015007	0.090116	0.089590	0.236202	0.215778
3	0.043918	0.048899	0.161593	0.145443	0.000855	0.001131	0.016893	0.019620	0.106270	0.102041
4	0.009981	0.013347	0.073066	0.072798	0.000039	0.000071	0.002375	0.003612	0.035859	0.040499
5	0.001736	0.003232	0.026430	0.032540	0.000001	0.000004	0.000267	0.000587	0.009680	0.014335
6	0.000252	0.000709	0.007967	0.013396	0.000000	0.000000	0.000025	0.000086	0.002178	0.004641
7	0.000031	0.000142	0.002058	0.005134	0.000000	0.000000	0.000002	0.000011	0.000420	0.001387
8	0.000003	0.000026	0.000465	0.001837	0.000000	0.000000	0.000000	0.000001	0.000071	0.000384
9	0.000000	0.000004	0.000094	0.000614	0.000000	0.000000	0.000000	0.000000	0.000011	0.000099
10	0.000000	0.000001	0.000017	0.000193	0.000000	0.000000	0.000000	0.000000	0.000001	0.000024
11	0.000000	0.000000	0.000003	0.000057	0.000000	0.000000	0.000000	0.000000	0.000000	0.000005
12	0.000000	0.000000	0.000000	0.000016	0.000000	0.000000	0.000000	0.000000	0.000000	0.000001
13	0.000000	0.000000	0.000000	0.000004	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
14	0.000000	0.000000	0.000000	0.000001	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
15	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
EBO	0.288633	0.306121	0.432787	0.485504	0.015825	0.017499	0.132234	0.142520	0.609040	0.633982

Execution time: METRIC = 0.08 sec.

EXACT = 2.54 sec.

Table III

The Probability Distribution of the Number of Units in Resupply at the Bases

$r_j = 0.90,\ j = 1,\ldots,5$

$s_1 = 1,\ s_2 = 1,\ s_3 = 1,\ s_4 = 1,\ s_5 = 1$

$\lambda_0 = 0.03897,\ S_0 = 2$

x	Base 1		Base 2		Base 3		Base 4		Base 5	
	METRIC	EXACT	METRIC	EXACT	METRIC	EXACT	METRIC	EXACT	METRIC	EXACT
0	0.555651	0.559602	0.29453	0.303458	0.883461	0.883744	0.683802	0.685849	0.401636	0.408461
1	0.326509	0.320993	0.360027	0.353315	0.109468	0.108938	0.259904	0.256612	0.366376	0.359104
2	0.095731	0.095837	0.220041	0.213715	0.006782	0.006995	0.049393	0.049995	0.167106	0.164188
3	0.018790	0.019869	0.089656	0.089733	0.000280	0.000312	0.006258	0.006763	0.050812	0.052125
4	0.002760	0.003212	0.027398	0.029414	0.000009	0.000011	0.000595	0.000713	0.011588	0.012914
5	0.000324	0.000430	0.006698	0.008009	0.000000	0.000000	0.000045	0.000062	0.002114	0.002655
6	0.000032	0.000050	0.001365	0.001880	0.000000	0.000000	0.000003	0.000005	0.000321	0.000470
7	0.000003	0.000005	0.000238	0.000389	0.000000	0.000000	0.000000	0.000000	0.000042	0.000073
8	0.000000	0.000000	0.000036	0.000072	0.000000	0.000000	0.000000	0.000000	0.000005	0.000010
9	0.000000	0.000000	0.000005	0.000012	0.000000	0.000000	0.000000	0.000000	0.000000	0.000001
10	0.000000	0.000000	0.000001	0.000002	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
EBO	0.143266	0.147217	0.516894	0.525817	0.007369	0.007652	0.063889	0.065935	0.313845	0.320669

Execution time: METRIC = 0.91 sec.

EXACT = 2.26 sec.

underestimates the stationary probability of zero unit in resupply (s_j units on hand at base j). Also, in the EXACT model, the stationary distribution of the number of units in resupply has longer tails. Furthermore, we wish to investigate how well the METRIC model approximates the stationary distribution when the depot spare stock level (S_0) changes. A similar investigation will be made when r_j, the probability that a failed unit is repairable at base j, changes. For this purpose, we arbitrarily examine base 2.

Figures 6.1, 6.2 and 6.3 show the probability distribution of the number of units in resupply at base 2 when $S_0 = 15$, 19 and 23, respectively. The value of r_2 is fixed at 0.10. It is clear from these figures that the discrepancy between the METRIC model and the EXACT model decreases as the value of S_0 increases. This can be explained as follows. As S_0 increases, in Eq. (4.25), the contribution of the term

$$\sum_{d_0(t_2,t_3)=S_0+d}^{\infty} \left\{ \binom{d_0(t_2,t_3)-S_0}{d} \cdot [\lambda_j^0/\lambda_0]^d [1-\lambda_j^0/\lambda_0]^{d_0(t_2,t_3)-d-S_0} \right\}$$

$\cdot\, P[d_0(t_2,t_3)|\lambda_0 R_0]$, $d \geq 0$ decreases and $\Pr\{B_j(*) = b\}$ is dominated by the Poisson term $P[s_j+b|\lambda_j^B R_j + \lambda_j^0 \tau_j] \cdot \left\{ \sum_{d_0(t_2,t_3)=0}^{S_0} P[d_0(t_2,t_3)|\lambda_0 R_0] \right\}$.

For the METRIC model, as S_0 increases the term $\delta(S_0) \cdot R_0$ representing the average delay per depot demand decreases and consequently $T_j(S_0)$, the average responce time, decreases. In Eq. (6.2), $\Pr\{B_j(*) = b\}$ is dominated by the Poisson term $P[s_j+b|\lambda_j^B R_j + \lambda_j^0 \tau_j]$. Thus we conclude that for the values of S_0 sufficiently large ($> \lambda_0 R_0$), the Poisson approximation is close to the results given by EXACT model. As $S_0 \to \infty$, $T_j(s_0) = \lambda_j^B R_j + \lambda_j^0 \tau_j$. Thus, the two models are the same and $\Pr\{B_j(*) = b\} = P[s_j+b|\lambda_j^B R_j + \lambda_j^0 \tau_j\}$.

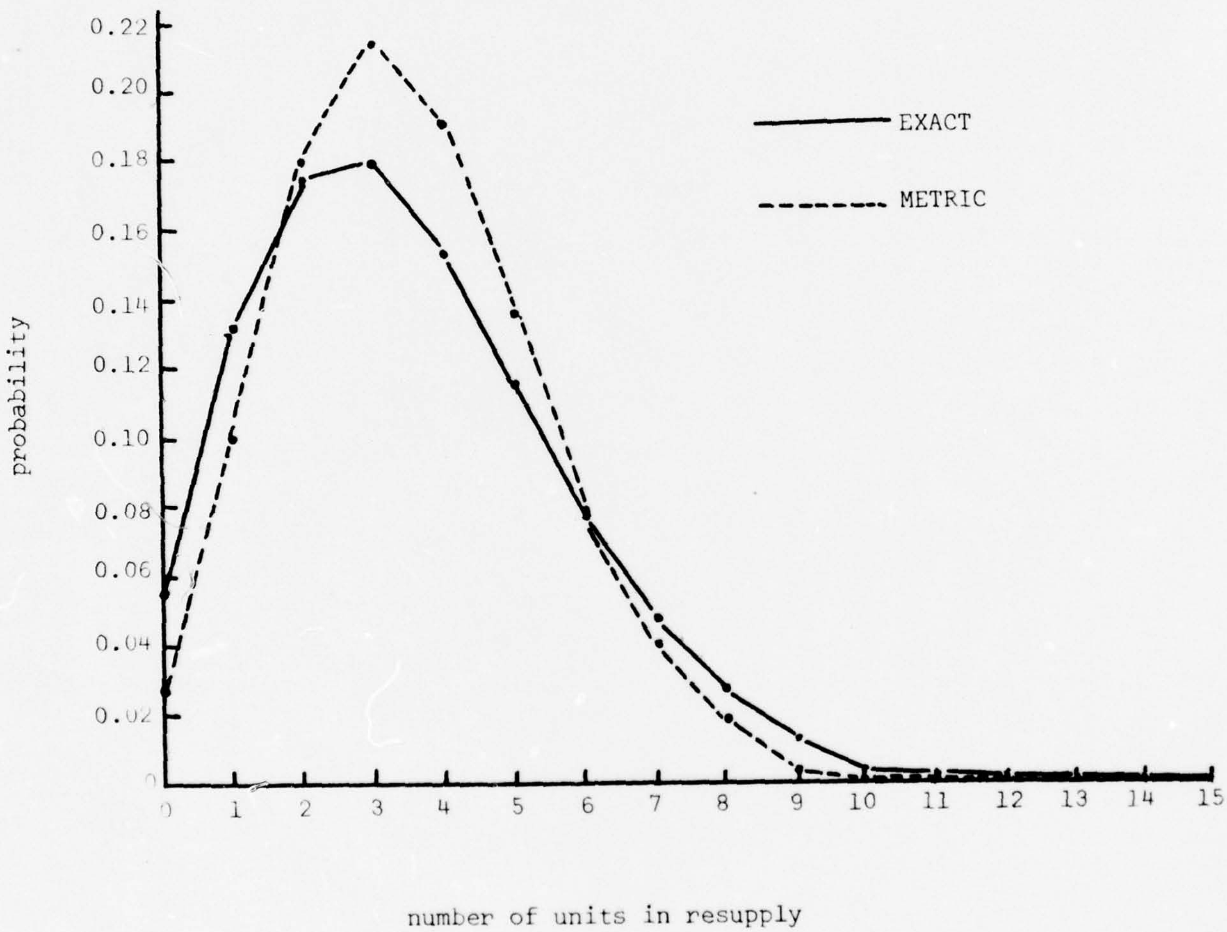


Figure 6.1: The probability distribution of the number of units in resupply at base 2 (r_2 = 0.10, S_0 = 15).

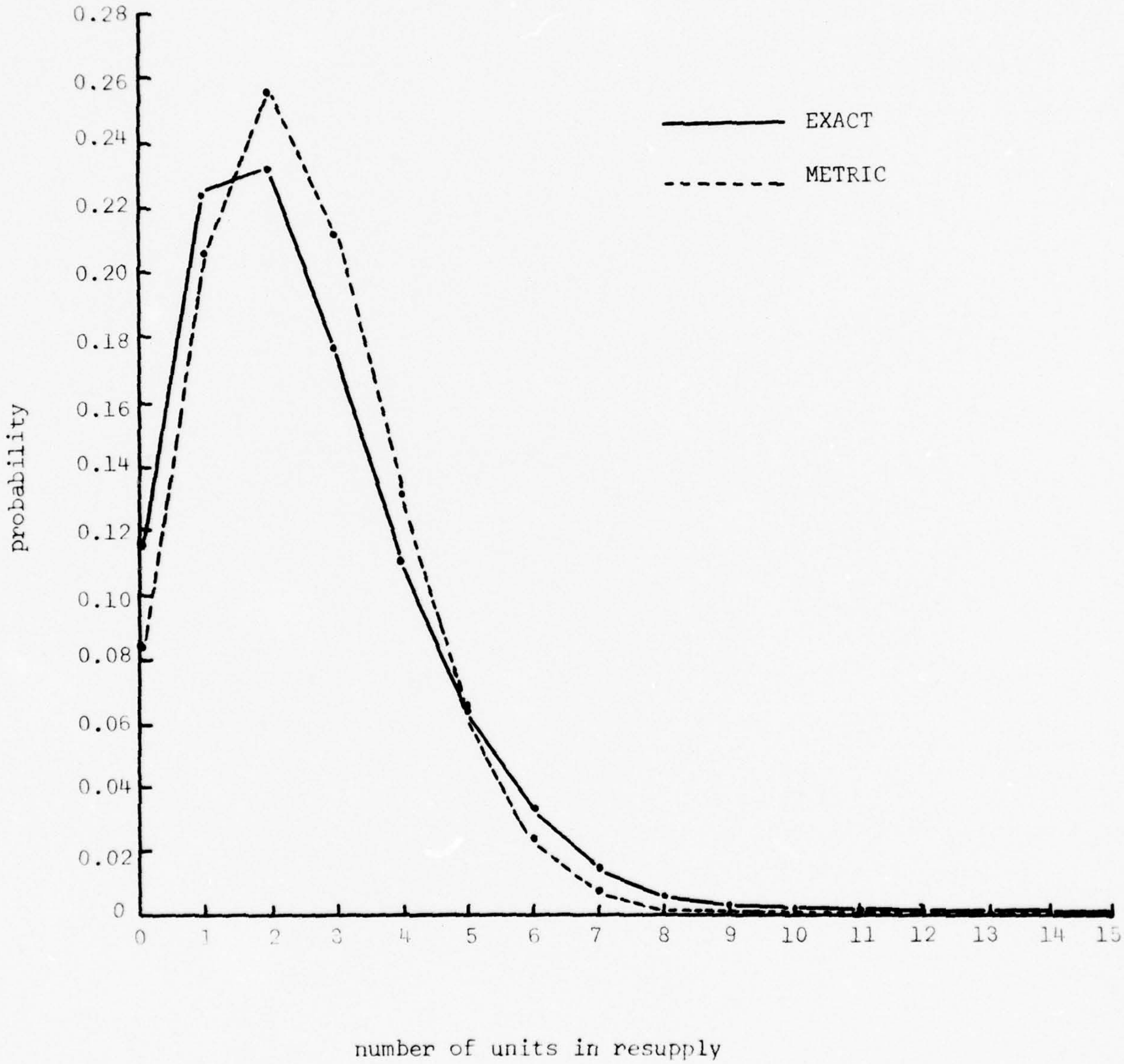


Figure 6.2: The probability distribution of the number of units in resupply at base 2 (r_2 = 0.10, S_0 = 19).

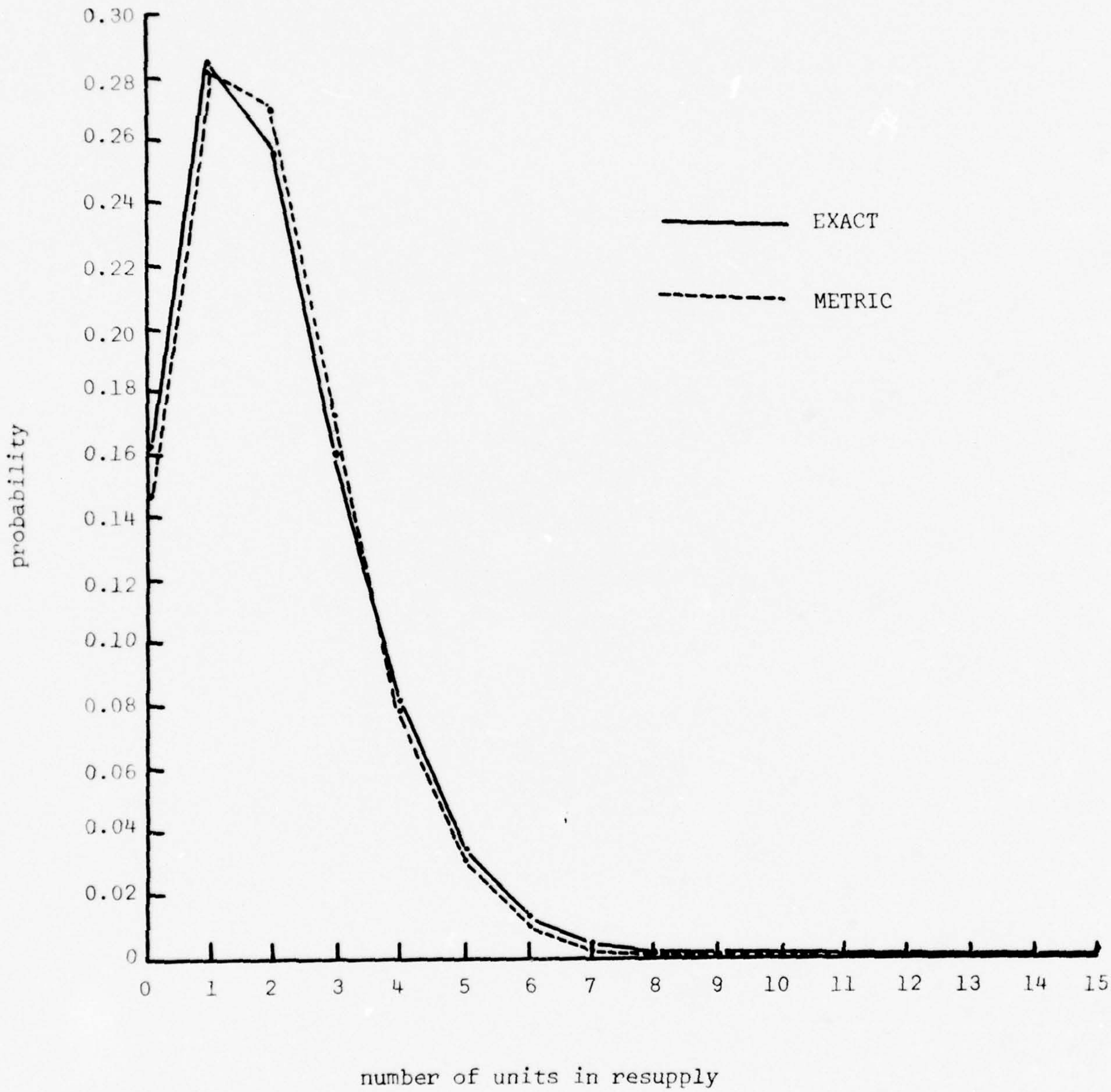


Figure 6.3: The probability distribution of the number of units in resupply at base 2 (r_2 = 0.10, S_0 = 23).

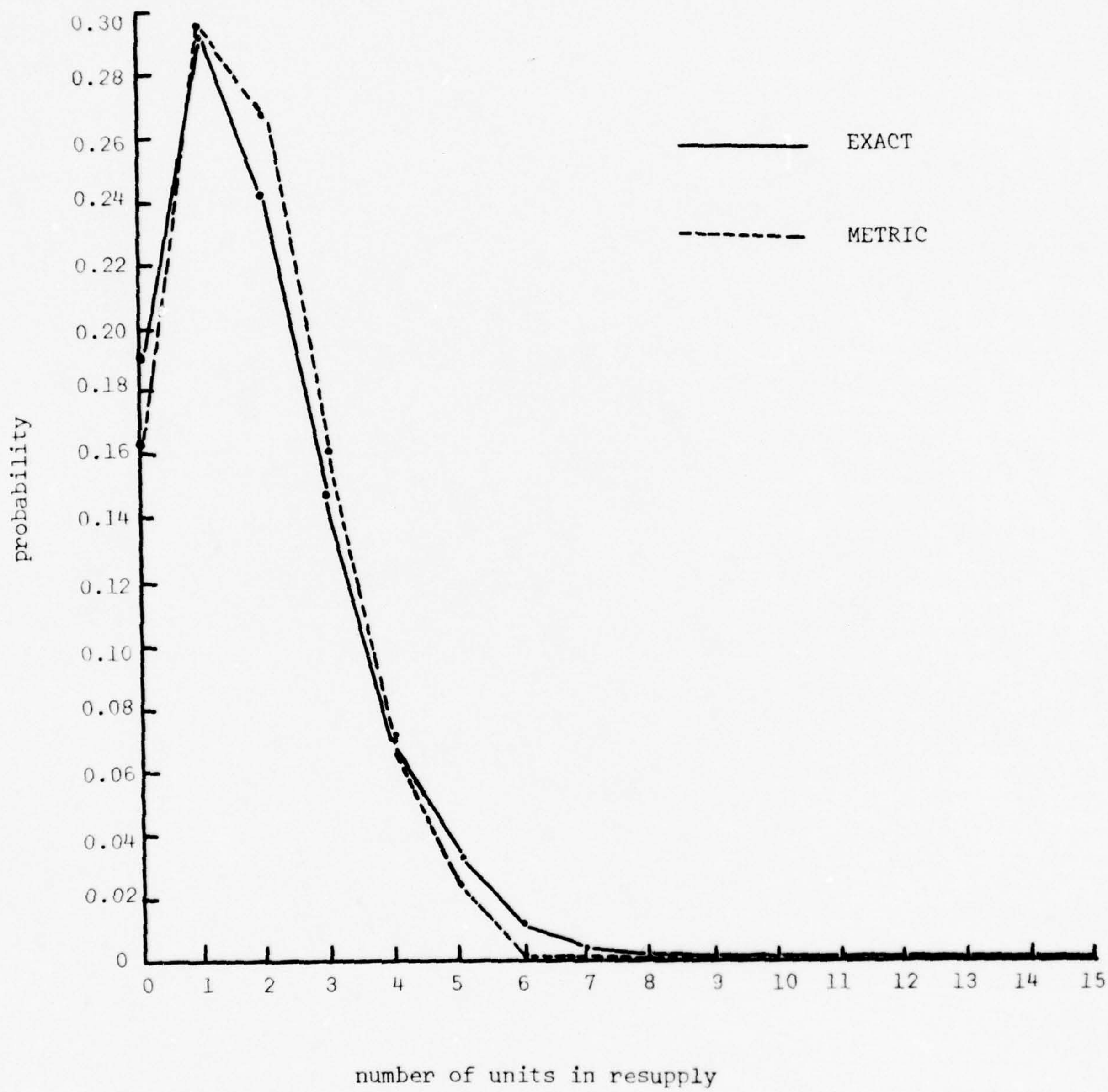


Figure 6.4: The probability distribution of the number of units in resupply at base 2 (r_2 = 0.5, S_0 = 11).

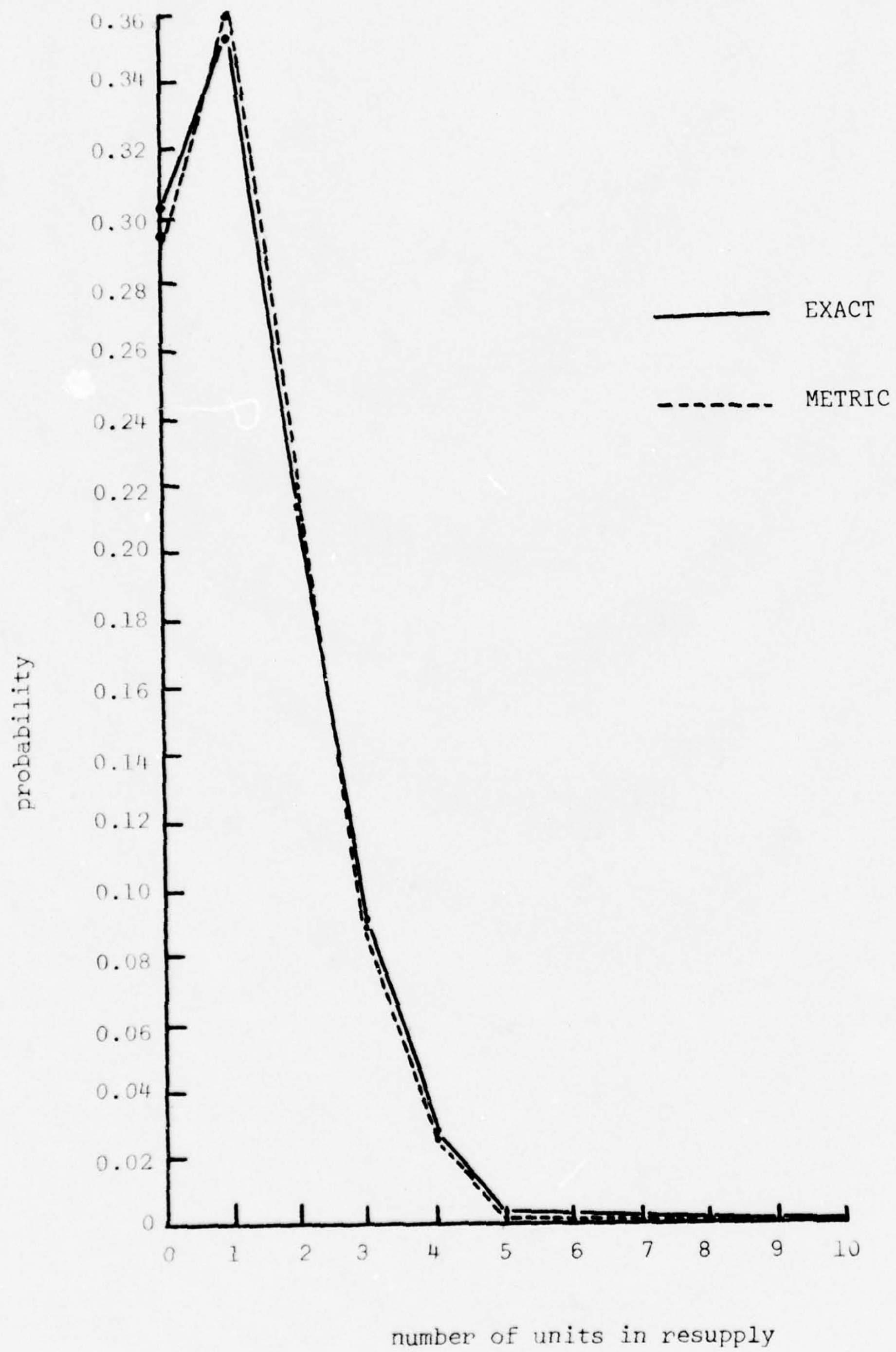


Figure 6.5: The probability distribution of the number of units in resupply at base 2 ($r_2 = 0.90$, $S_0 = 2$).

We now investigate how well the METRIC model approximates the EXACT model when r_j, changes. Figures 6.2, 6.4 and 6.5 show the probability distribution of the number of units in resupply at base 2 for the case when $r_2 = 0.10$, $r_2 = 0.50$ and $r_2 = 0.90$, respectively. The depot spare stock level for these cases is fixed at 19,11 and 2, respectively. The stock level is approximately equal to the corresponding value of $\lambda_0 R_0$ in each case. It is clear from Figures 6.2, 6.4 and 6.5 that the discrepancy between the METRIC and the EXACT model decreases as r_2 increases. This can be explained in a way similar to the previous case of comparing the two models with respect to S_0. When r_j increases λ_j^B increases and λ_j^0 decreases. Consequently, the proportion of supply received at the base from its repair process increases. Thus, in both Eqs. (4.25) and (6.2), $Pr\{B_j(*) = b\}$ is dominated by the Poisson term $P[s_j+b|\lambda_j^B R_j]$. As $r_j \to 1$, both Eqs. (4.25) and (6.2) reduce to $P[s_j+b|\lambda_j R_j]$ and the two models are the same.

As mentioned by Sherbrooke [17], the expected number of backorders at the bases (EBO) is an important measure of system performance. We must make an assessment of how well the METRIC model approximates the computation of EBO. From tables I,II and III it is clear that the METRIC model always gives the values of EBO which are less than or equal to the corresponding values given by the EXACT model. Furthermore, it is clear from Figures (6.1-6.5) that the METRIC curve is always below the EXACT curve near the tail of the distribution. In other words, the METRIC model always underestimates the probability for the larger numbers of units in resupply at a base. Consequently, as the base spare stock level is increased the difference between the values of EBO given by the two models increases. Table IV displays such a discrepancy for base 2 of the example.

Table IV

The Expected Number of Backorders at Base 2

$r_2 = 0.50$, $S_0 = 11$

s_2	METRIC	EXACT
0	1.808638	1.808638
1	0.972516	0.999878
2	0.432787	0.485504
3	0.161094	0.213471
4	0.050994	0.086882
5	0.013959	0.033091
6	0.003354	0.011839
7	0.000717	0.003984
8	0.000138	0.001262
9	0.000024	0.000377

Finally, the EXACT model requires more execution time than the METRIC model. This is so because in the EXACT model, the expression for $\Pr\{B_j(*) = b\}$ is relatively complex. It involves the computations of the combinatorial terms in addition to the Poisson terms, and a large number of calculations are required to evaluate $\Pr\{B_j(*) = b\}$.

We conclude this chapter by emphasizing that although the METRIC model is computationally simple, its use in certain situations may not be appropriate. When the depot spare stock level S_0 is low (less than $\lambda_0 R_0$, the mean demand during the repair time) and the proportion of repairs done at the depot is high (r_j is low), there is a pronounced difference between the two models. Thus in this situation the METRIC results may be misleading. For higher values of S_0 and r_j, however, the approximate results given by the METRIC model are fairly close to the results given by the EXACT model. Furthermore, when an optimization algorithm is applied to determine the optimal stock levels of several items at different locations in the system under a budget constraint, the solution may be wrong if the METRIC results are used. This may happen because of the above mentioned varying discrepancy between the two models with respect to S_0 and r_j. Suppose the problem is to determine the optimal stock levels for two items, for one of which the METRIC and the EXACT results are close while for the other there is a considerable difference between the results given by the two models. In other words, for the second item the METRIC curve is steeper than the EXACT curve towards the tail of the distribution of the number of units in resupply. Thus the reduction in the expected number of backorders per additional unit

of the second item is larger for the METRIC model. Consequently, it will erroneously result in a larger appropriation of spare stocks for the second item if the METRIC results are used. A similar conclusion can be drawn for the case when the problem is to allocate optimally the spare stocks of a single item at different bases in the system. Here, it easily follows from the above arguments that for the bases at which the repair capability is poor (r_j is low), the METRIC model will suggest larger spare stock levels than those given by the EXACT model.

CHAPTER VII

CONCLUDING COMMENTS

In this study, we have presented an analysis for continuous review models of single location and two-echelon inventory systems for recoverable items with random demands.

For the single location system, the (s,S) and (s,nQ) policies have been examined. Under each policy, stationary distributions for inventory position, on-hand inventory and backorders have been obtained for both independent and dependent demand processes. For the diagnosis of failed units two appealing models - batch and unit - were examined. Our analysis is more general than any previous work since only independent demand processes and batch inspection were considered before. We have also extended our results to the case where demand processes are compound Poisson processes. In addition, we have demonstrated how the special cases of complete recoverability and complete non-recoverability are obtained from our results with a simple change of parameters.

As mentioned in Section 3.1, the analysis of a single location system, in addition to offering the solution to the system itself, is of great importance for the analysis of multi-echelon systems. The results that we obtained do not offer a complete solution to the complex problems facing inventory managers in such a system. Using the approach and results obtained here, further analysis may be pursued to evaluate cost/benefit trade-offs among the various system parameters and to examine alternative repair disciplines.

For the two-echelon system, we have mainly concentrated on obtaining an exact expression for stationary distribution of the number of backorders at each base. The approach suggested by Kruse and Kaplan [9] has been extended to the case of an arbitrary order size distribution at the bases. Here, too, the batch and unit inspection models have been considered. The results have been indicated for the two extreme cases of complete recoverability and complete non-recoverability of the item. For the case of complete recoverability, we have demonstrated how the METRIC model may produce a misleading solution to the problem of allocating items in the system.

Our results for the two-echelon system depend heavily on the assumption that failures are generated by Poisson processes, repair and lead times are constant, and partial backlogging is allowed. Also, we have considered and (s-1,s) policy at the bases and (s,S) policy at the depot. We have assumed that a first-come, first-served policy is used at all locations to fill the backlogged demands. We make no claim that these assumptions are always true. Nevertheless, our approach should be useful as a basis for further analysis.

The methodology developed in chapter IV and V may be used to obtain the stationary distribution of the number of backorders when an (r,Q) or an (s,S) policy is used at the bases. It will be interesting to examine the situation where the inter-arrival time at each base has an Erlang distribution. The Erlang distribution, as mentioned by Gross and Harris ([6], p. 162-163), provides much more modeling flexibiltty than does the exponential. It can be used to provide good approximation for many different inter-arrival distributions.

An attempt may be made to analyze the situation where the depot does not strictly follow the first-come, first-served policy to fill the base demands. On the other hand, a priority scheme is followed at the depot such that the bases with low mean demand would have priority over the bases with high mean demand. Certainly, this will not affect the total expected number of system backorders, but will result in lower spare stock levels at the low demand bases.

Finally, our approach can be directly used to analyze arborescence systems with more than two echelons where all the bases and intermediate echelons use an $(s-1,s)$ policy and the depot uses an (s,S) policy. In addition to bases and the depot, repair can be performed at intermediate echelons.

APPENDIX A

ARRIVAL SEQUENCES

In this appendix, we derive some results concerning the sequence of arrivals generated by p (≥ 2) independent Poisson processes. We first consider the case when $p = 2$. Let $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ be the two independent Poisson processes with parameters λ_1 and λ_2, respectively. Also, let

$N(t)$ = total number of arrivals during the interval $(0,t]$;

T_k^i = the time of the k^{th} arrival from process i, $i = 1,2$ and $k \geq 1$ ($T_0^i = 0$, $i = 1,2$);

and

W_x = the time of the x^{th} arrival in the total arrival process $\{N(t), t \geq 0\}$.

We have the following theorem.

Theorem 1. Let n_1 and n_2 be non-negative integers such that $n_1 + n_2 > 0$. Also, suppose we are given non-negative integers n_1' and n_2' such that

(a) $n_1' \leq n_1' + n_2' < n_1 + n_2$

(b) $n_1' + n_2' > 0$.

Then

$$\text{(A.1)} \qquad \Pr\{N_2(T^1_{n_1'}) = n_2' | N_1(t) = n_1;\ N_2(t) = n_2\}$$

$$= \frac{n_1'}{n_1'+n_2'} \cdot \frac{\binom{n_1}{n_1'}\binom{n_2}{n_2'}}{\binom{n_1+n_2}{n_1'+n_2'}} .$$

Proof: We know that $T^1_{n_1'}$, the time of $n_1'^{\text{th}}$ arrival from process 1, has a gamma distribution with parameters n_1' and λ_1 (see Ross [14], p. 16-17); that is, the probability density function of $T^1_{n_1'}$ is given by

$$\text{(A.2)} \qquad f(T^1_{n_1'}, u) = \frac{e^{-\lambda_1 u}(\lambda_1)^{n_1'}(u)^{n_1'-1}}{(n_1'-1)!} \qquad 0 \le u \le t.$$

We can write

$$\text{(A.3)} \qquad \Pr\{N_2(T^1_{n_1'}) = n_2' | N_1(t) = n_1;\ N_2(t) = n_2\}$$

$$= \frac{\Pr\{N_2(T^1_{n_1'}) = n_2';\ N_1(t) = n_1;\ N_2(t) = n_2\}}{\Pr\{N_1(t) = n_1;\ N_2(t) = n_2\}}$$

The numerator of Eq. (A.3) can be evaluated by conditioning on $T^1_{n_1'}$. Thus we have,

$$\text{(A.4)} \quad \Pr\{N_2(T^1_{n_1'}) = n_2';\ N_1(t) = n_1;\ N_2(t) = n_2\}$$

$$= \int_0^t \Pr\{N_2(u) = n_2';\ N_1(t) = n_1;\ N_2(t) = n_2 | T^1_{n_1'} = u\} \cdot f(T^1_{n_1'}, u)$$

$$= \int_0^t \Pr\{N_2(u) = n_2';\ N_1(t-u) = n_1 - n_1';\ N_2(t-u) = n_2 - n_2'\} \cdot f(T^1_{n_1'}, u)$$

$$= \int_0^t \frac{e^{-\lambda_2 u}(\lambda_2 u)^{n_2'}}{n_2'!} \cdot \frac{e^{-\lambda_2(t-u)}[\lambda_1(t-u)]^{n_1-n_1'}}{(n_1-n_1')!} \cdot \frac{e^{-\lambda_2(t-u)}[\lambda_2(t-u)]^{n_2-n_2'}}{(n_2-n_2')!} \cdot$$

$$\frac{e^{-\lambda_1 u}(\lambda_1)^{n_1'}(u)^{n_1'-1}}{(n_1'-1)!}\, du$$

$$= \frac{e^{-\lambda_1 t}\lambda_1^{n_1} e^{-\lambda_2 t}\lambda_2^{n_2}}{(n_1'-1)!\ n_2'!\ (n_1-n_1')!\ (n_2-n_2')!} \int_0^t u^{n_1'+n_2'-1} (t-u)^{n_1+n_2-n_1'-n_2'} du$$

$$= \frac{e^{-(\lambda_1+\lambda_2)t} (\lambda_1)^{n_1} (\lambda_2)^{n_2}}{(n_1'-1)!\ n_2'!\ (n_1-n_1')\ !\ (n_2-n_2')!} \cdot t^{n_1+n_2} \cdot \frac{(n_1'+n_2'-1)!\ (n_1+n_2-n_1'-n_2')!}{(n_1+n_2)!} .$$

We know for the denonimator of Eq. (A.3) that

$$\text{(A.5)} \qquad Pr\{N_1(t) = n_1;\ N_2(t) = n_2\} = \frac{e^{-\lambda_1 t}(\lambda_1 t)^{n_1}}{n_1!} \cdot \frac{e^{-\lambda_2 t}(\lambda_2 t)^{n_2}}{n_2!} .$$

Substituting Eq. (A.4) and (A.5) into (A.3) and upon simplifying we obtain Eq. (A.1).

Q.E.D.

We now have the following two corollaries.

Corollary 1.1 Under the assumptions of Theorem 1, let x, y be non-negative integers such that $y \leq x < n_1 + n_2$ and $x > 0$. Then

$$\text{(A.6)} \qquad Pr\{N_1(W_x) = y | N_1(t) = n_1;\ N_2(t) = n_2\}$$

$$= \frac{\binom{n_1}{y}\binom{n_2}{x-y}}{\binom{n_1+n_2}{x}}$$

Proof: We note that the event $N_1(W_x) = y$ is equivalent to the event $N_2(W_x) = x-y$. The waiting time W_x will coincide either with T_y^1 or T_{x-y}^2. Therefore,

$$\Pr\{N_1(W_x) = y | N_1(t) = n_1;\ N_2(t) = n_2\}$$

$$= \Pr\{N_2(T_y^1) = x-y | N_1(t) = n_1;\ N_2(t) = n_2\}$$

$$+ \Pr\{N_1(T_{x-y}^2) = x | N_1(t) = n_1;\ N_2(t) = n_2\}$$

$$= \left\{\frac{y}{x} + \frac{x-y}{x}\right\} \cdot \frac{\binom{n_1}{y}\binom{n_2}{x-y}}{\binom{n_1+n_2}{x}} \quad \text{(from Theorem 1)}$$

$$= \frac{\binom{n_1}{y}\binom{n_2}{x-y}}{\binom{n_1+n_2}{x}} .$$

Q.E.D.

The above corollary has been also proved by Simon [20] under more general conditions.

Corollary 1.2. Let $I(s)$ denote the index of the process from which the next arrival occurs after time s $(0 \leq s < t)$. Then under the assumptions of Theorem 1,

(A.7) $$\Pr\{I(W_x) = i | N_i(W_x) = n_i',\ N_1(t) = n_1;\ N_2(t) = n_2\}$$

$$= \frac{n_i - n_i'}{(n_1+n_2) - x} \qquad i = 1,2.$$

Proof: We prove for the case when $i = 1$. Given that $N_1(t) = n_1$, $N_2(t) = n_2$ and $N_1(W_x) = n_1'$, the event $I(W_x) = 1$ means that the next arrival after n_1' arrivals from process 1 and $(x-n_1')$ arrivals

from process 2 is from process 1. In other words, by the time $(n_1'+1)^{th}$ arrival from process 1 occurs, there are $(x-n_1')$ arrivals from process 2. Thus we have,

$$\Pr\{I(W_x) = 1 | N_1(W_x) = n_1';\ N_1(t) = n_1;\ N_2(t) = n_2\}$$

$$= \frac{\Pr\{I(W_x) = 1;\ N_1(w_x) = n_1' | N_1(t) = n_1;\ N_2(t) = n_2\}}{\Pr\{N_1(W_x) = n_1' | N_1(t) = n_1;\ N_2(t) = n_2\}}$$

The numerator of the above expression is equivalent to

$$\text{(A.8)}\quad \Pr\{N_2(T^1_{(n_1'+1)}) = x-n_1' | N_1(t) = n_1;\ N_2(t) = n_2\}$$

$$= \frac{n_1'+1}{x+1}\cdot\frac{\binom{n_1}{n_1'+1}\binom{n_2}{x-n_1'}}{\binom{n_1+n_2}{x+1}} \quad \text{(from Theorem 1)}$$

$$= \frac{n_1-n_1'}{n_1+n_2-x}\cdot\frac{\binom{n_1}{n_2}\binom{n_2}{x-n_1'}}{\binom{n_1+n_2}{x}}$$

From Corollary 1.2, we have

$$\text{(A.9)}\quad \Pr\{N_1(W_x) = n_1' | N_1(t) = n_1;\ N_2(t_2) = n_2\}$$

$$= \frac{\binom{n_1}{n_1'}\binom{n_2}{x-n_1'}}{\binom{n_1+n_2}{x}}\ .$$

From Eqs. (A.8) and (A.9) we obtain Eq. (4.7)

Q.E.D.

We now extend the above results to the case when there are more than two independent sources generating the arrivals. Let $\{N_i(t), t \geq 0\}$, $i = 1,2,\ldots,p$ be the independent Poisson processes with parameter λ_i, respectively. We use the following notation similar to those used in the previous case.

$$N(t) = \sum_{i=1}^{p} N_i(t), \quad t \geq 0.$$

T_k^i = the time of the k^{th} arrival from process i, $i = 1,2,\ldots,p$ and $k \geq 1$, $(T_0^i = 0, \ i = 1,2,\ldots,p)$.

W_x = the time of the x^{th} arrival in the total arrival process $\{N(t), t \geq 0\}$.

Similar to theorem 1, we have

Theorem 2. Let $\{n_i\}$, $\{n_i'\}$ $i=1,2,\ldots,p$ be non-negative integers such that

(a) $\sum_{i=1}^{p} n_i > 0, \quad \sum_{i=1}^{p} n_i' > 0$

and

(b) $n_i' \leq \sum_{i=1}^{p} n_i' < \sum_{i=1}^{p} n_i \quad i = 1,2,\ldots,p;$

Then

(A.10) $\Pr\{N_j(T^i_{n_i'}) = n_j',\ j = 1,2,\ldots,p;\ j \neq i \mid N_k(t) = n_k,\ k = 1,2,\ldots,p\}$

$$= \frac{n_i}{\sum_{j=1}^{p} n_j'} \frac{\prod_{j=1}^{p} \binom{n_j}{n_j'}}{\binom{\sum_{j=1}^{p} n_j}{\sum_{j=1}^{p} n_j'}}$$

Proof: Similar to the proof of theorem 1.2.

Parallel to Corollaries 1.1 and 1.2 we have the following.

Corollary 2.1. Under the assumptions of Theorem 2,

$$\Pr\{N_i(W_{(n_1+n_2+\ldots n_p')}) = n_i',\ i = 1,2,\ldots,p \mid N_i(t) = n_i,\ i = 1,2,\ldots,p\}$$

$$= \frac{\prod_{i=1}^{p} \binom{n_i}{n_i'}}{\binom{\sum_{i=1}^{p} n_i}{\sum_{i=1}^{p} n_i'}}$$

Proof: Similar to the proof of Corollary 1.1.

Corollary 2.2. Let $I(s)$ denote the index of the process from which the next arrival occurs after time s ($0 \leq s < t$), then

$$\Pr\{I(W_x) = i \mid N_i(W_x) = n_i',\ N_k(t) = n_k,\ k = 1,2,\ldots,p\}$$

$$= \frac{n_i - n_i'}{\sum_{i=1}^{p} n_j - \sum_{j=1}^{p} n_j'} \qquad i = 1,2,\ldots,p.$$

Proof: Similar to the proof of Corollary 1.2.

APPENDIX B

SIMPLIFICATION OF COMBINATORIAL EXPRESSIONS

1. Derivation of Eq. (4.19)

We first simplify Eq. (4.16). To guarantee that all values within the combinatorial expressions are non-negative, the range of $d_j^0(t_2,t_3)$ satisfies: $0 \leq d_1^0(t_2,t_3) \leq z_0(t_1) - d_0^C(t_1,t_2)$. Expanding the combinatorial expressions and simplifying the summation over $d_j^0(t_2,t_3)$, we obtain

$$\sum_{d_j^0(t_2,t_3)=0}^{z_0(t_1)-d_0^C(t_1,t_2)} \left\{ \binom{z_0(t_1)-d_0^C(t_1,t_2)}{d_j^0(t_2,t_3)} [\lambda_j^0/\lambda_0]^{d_j^0(t_2,t_3)} [1-\lambda_j^0/\lambda_0]^{d_0(t_2,t_3)-d_j^0(t_2,t_3)} \right\}$$

$$= [1-\lambda_j^0/\lambda_0]^{d_0(t_2,t_3)-z_0(t_1)+d_0^C(t_1,t_2)}$$

$$\cdot \left[\sum_{d_j^0(t_2,t_3)=0}^{z_0(t_1)-d_0^C(t_1,t_2)} \left\{ \binom{z_0(t_1)-d_0^C(t_1,t_2)}{d_j^0(t_2,t_3)} [\lambda_j^0/\lambda_0]^{d_j^0(t_2,t_3)} [1-\lambda_j^0/\lambda_0]^{z_0(t_1)-d_0^C(t_1,t_2)-d_j^0(t_2,t_3)} \right\} \right]$$

$$= [1-\lambda_j^0/\lambda_0]^{d_0(t_2,t_3)-z_0(t_1)+d_0^C(t_1,t_2)}$$

Similarly, to simplify Eq. (4.17), we find that the ranges of

$d_j^0(t_2,t_3)$ and $d_0(t_2,t_3)$ yielding non-negative values within the combinatorial expressions are

$$\text{(i)} \quad d \leq d_j^0(t_2,t_3) \leq z_0(t_1) - d_0^C(t_1,t_2) + d,$$

$$\text{and} \quad \text{(ii)} \qquad d_0(t_2,t_3) \geq z_0(t_1) - d_0^C(t_1,t_2) + d.$$

Expanding the combinatorial expressions and simplifying we obtain

$$\sum_{d_j^0(t_2,t_3)=d}^{z_0(t_1)-d_0^C(t_1,t_2)+d} \left\{ \binom{d_0(t_2,t_3) - z_0(t_1) + d_0^C(t_1,t_2)}{d} \circ \binom{z_0(t_1) - d_0^C(t_1,t_2)}{d_j^0(t_2,t_3)} \cdot [\lambda_j^0/\lambda_0]^{d_j^0(t_2,t_3)+d} \circ [1 - \lambda_j^0/\lambda_0]^{d_0(t_2,t_3)-d_j^0(t_2,t_3)-d} \right\}$$

$$= \binom{d_0(t_2,t_3) - z_0(t_1) - d_0^C(t_1,t_2)}{d} \cdot [\lambda_j^0/\lambda_0]^d [1-\lambda_j^0/\lambda_0]^{d_0(t_2,t_3) - z_0(t_1) - d + d_0^C(t_1,t_2)} \cdot \left\{ \sum_{d_j^0(t_2,t_3) = 0}^{z_0(t_1) - d_0^C(t_1,t_2)} \left\{ \binom{z_0(t_1) - d_0^C(t_1,t_2)}{d_j^0(t_2,t_3)} \cdot [\lambda_j^0/\lambda_0]^{d_j(t_2,t_3)} [1-\lambda_j^0/\lambda_0]^{z_0(t_1) - d_0^C(t_1,t_2) - d_j^0(t_2,t_3)} \right\} \right\}$$

$$= \binom{d_0(t_2,t_3) - z_0(t_1) - d_0^C(t_1,t_2)}{d}$$

$$[\lambda_j^0/\lambda_0]^d [1-\lambda_j^0/\lambda_0]^{d_0(t_2,t_3)-z_0(t_1)-d+d_0^C(t_1,t_2)}$$

Substituting these simiplifications and taking the limit as $t \to \infty$, we obtain Eq. (4.19).

2. Derivation of Eq. 4.24)

From Eqs. (4.21) and (4.23) we obtain the ranges of $d_j^0(t_1,t_2)$ and $d_0(t_1,t_2)$ such that the values within the combinatorial expressions are nonnegative.

$$\text{(i)} \quad d \leq d_j^0(t_1,t_2) \leq z_0(t_1) + d_0^D(t_1,t_2) + d$$

$$\text{and} \quad \text{(ii)} \quad d_0^C(t_1,t_2) \geq z_0(t_1) + d.$$

Upon expanding the combinatorial expressions and simplifying in the manner similar to that in the previous case we obtain Eq. (4.24).

BIBLIOGRAPHY

[1] S.G. Allen and D.A. D'Esopo, "An Ordering Policy for Repairable Stock Items," Opns. Res. 16, p. 669-774 (1968).

[2] P. Billingsley. Convergence of Probability Measures, Wiley, New York (1968).

[3] A.J. Clark, "An Informal Survey of Multi-Echelon Inventory Theory", Naval Res. Log. Quart., 19, p. 621-650 (1972).

[4] G.F. Feeney and C.C. Sherbrooke, "A System Approach to Base Stockage of Recoverable Items", The Rand Corporation, RM-4720-PR (DDC No. AD 626744). (December 1965).

[5] G.F. Feeney and C.C. Sherbrooke, "The (s-1, s) Inventory Policy Under Compound Poisson Demand," Mgmt. Sc., 12, p. 391-411 (1966).

[6] D. Gross and C.M. Harris, Fundamentals of Queueing Theory, John Wiley and Sons (1974).

[7] G. Hadley and T.M. Whitin, Analysis of Inventory Systems, Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1963).

[8] D.L. Isaacson and R.W. Madsen, Markov Chains Theory and Applications , John Wiley (1975).

[9] W.K. Kruse and A.J. Kaplan, "On a Paper by Simon", Opns. Res., 21, p. 1318-1322 (1973).

[10] Francois Lureau, "A Queueing Theoretic Analysis of Logistics Repair Models with Spare Units", Technical Report No. 167, Department of Operations Research and Department of Statistics, Stanford University, Stanford, California (December, 1974).

[11] J.A. Muckstadt, "A Model for a Multi-Item, Multi-Echelon, Multi-Indenture Inventory System," Mgmt. Sc., 20, p. 472-481, (1973).

[12] J.A. Muckstadt, "Some Approximations in Multi-Item, Multi-Echelon Inventory Systems for Recoverable Items", The Rand Corporation, P-5763 (December 1976).

[13] F.R. Richards, "A Repairable Item Inventory System", Opns. Res., 24, p. 118-130 (1976).

[14] S.M. Ross, Applied Probability Models With Optimization Applications, Holden-Day (1970).

[15] D.A. Schrady, "A Deterministic Inventory Model for Repairable Items," Naval Res. Log. Quart. 14, p. 391-398 (1967).

[16] D.A. Schrady, "Mathematical Models of the Repairable Item Inventory System", Report No. NPS 55S07081A, U.S. Naval Postgraduate School, Monterey, California (August 1967).

[17] C.C. Sherbrooke, "METRIC: A Multi-Echelon Technique for Recoverable Item Control", Opns. Res., 16, p. 122-141 (1968).

[18] C.C. Sherbrooke, "Discrete Compound Poisson Processes and Tables of Geometric Poisson Distribution", The Rand Corporation, RM-4831-PR (July 1966).

[19] R.M. Simon, "The Uniform Distribution of Inventory Position for Continuous Review (s,Q) Policies", P-3938, The Rand Corp., Santa Monica, California (1968).

[20] R.M. Simon, "Stationary Properties of a Two-Echelon Inventory Model for Low-Demand Items", Opns. Res. 19, p. 761-773 (1971).

[21] R.M. Simon, and D.A. D'Esopo, "Comments on a Paper by S.G. Allen and D.A. D'Esopo: An Ordering Policy for Repairable Stock Items", Opns. Res., 19, p. 986-988 (1971).

[22] B.D. Sivazlian, "A Continuous Review (s,S) Inventory System with Arbitrary Interarrival Distribution between Unit Demand", Opns. Res. 22, p. 65-71 (1974).

[23] H.C. Tijms, Analysis of (s,S) Inventory Models, Mathematisch Centrum, Amsterdam (1972).

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE	READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER: 14 TR-341 / 2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): 6 AN ANALYSIS OF A TWO-ECHELON INVENTORY SYSTEM FOR RECOVERABLE ITEMS.	5. TYPE OF REPORT & PERIOD COVERED: 9 Technical Report,
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): 10 Kripa Shanker	8. CONTRACT OR GRANT NUMBER(s): 15 N0014-75-C-1172
9. PERFORMING ORGANIZATION NAME AND ADDRESS: School of Operations Research and Industrial Engineering, College of Engineering, Cornell University, Ithaca, New York 14853	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: Task NR 042-335
11. CONTROLLING OFFICE NAME AND ADDRESS: Sponsoring Military Activity, Statistics and Probability Program, Office of Naval Research, Arlington, VA 22217	12. REPORT DATE: 11 July 1977
	13. NUMBER OF PAGES: 165 12 167p.
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Inventory Theory	Arrival Sequences
Single Location System	Unit Model
Two-Echelon System	Batch Model

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this dissertation, we present an analysis of continuous review models of a two-echelon inventory system for recoverable items. The system consists of a depot and a set of bases. Primary demands occur at the bases for one or several units at a time. It is assumed that demands arrive in a Poisson manner. Upon arrival of a demand for certain units, a like number of failed units are turned in at the base. An inspection of the failed units is carried out to decide whether the units will be repaired at the base or at the depot or will →

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified 409869

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

be removed from the system in case repair is not economical. The bases use an (s-1, s) policy for procurement of serviceable units from the depot, and the depot uses an (s,S) policy to procure from the external supplier. Demands in an out-of-stock situation are backlogged. It is assumed that all the locations have infinite repair capacities and repair and procurement lead times are constant.

A common problem in inventory management is to specify the policy parameters that will minimize expected cost per unit time for operating the system subject to constraints of certain performance measures. To formulate such a problem we must find the stationary distributions for inventory position, on-hand inventory, backorders and in-repair inventory. Our main objective is to find exact expressions for these distributions.

The investigation begins with an extensive analysis of a single location system. The procurement policy is a continuous review (s,S) policy. The inter-arrival times between successive requisitions are independent and identically distributed random variables. The system experiences two types of demands - recoverable and non-recoverable. The two demand processes may be independent or dependent. For the inspection of failed units, two models - batch and unit - are considered. In the batch model, the entire batch is either recoverable or non-recoverable, whereas, in the unit model each unit in a batch is inspected independently. the special cases of compound Poisson demands, (s,nQ) procurement policy, complete recoverability and complete non-recoverability are also considered.

For the two-echelon system we first consider the case where demands at the bases occur for a single unit at a time. The approach is then applied to a general situation where demands at the bases are random. Both the batch and unit inspection models are considered. For the case when there are no condemnations of the item, results are compared with the METRIC model. The METRIC model provides a simple but approximate expression for the probability distribuation of system backorders. The comparison indicates that there is a considerable discrepancy between the METRIC results and our results when the depot spare stock is low or when a major proportion of the repair is done at the depot.